An *h*-narrow band finite-element method for elliptic equations on implicit surfaces

KLAUS DECKELNICK

*Institut für Analysis und Numerik, Otto-von-Guericke Universität Magdeburg,
Universitätsplatz 2, 39106 Magdeburg, Germany*

GERHARD DZIUK

*Abteilung für Angewandte Mathematik, Universität Freiburg, Hermann-Herder Strasse 10,
79104 Freiburg, Germany*

CHARLES M. ELLIOTT†

*Mathematics Institute and Centre for Scientific Computing, University of Warwick,
Coventry CV4 7AL, UK*

AND

CLAUS-JUSTUS HEINE

*Abteilung für Angewandte Mathematik, Universität Freiburg, Hermann-Herder Strasse 10,
79104 Freiburg, Germany*

In this article we define a finite-element method for elliptic partial differential equations (PDEs) on curves or surfaces, which are given implicitly by some level set function. The method is specially designed for complicated surfaces. The key idea is to solve the PDE on a narrow band around the surface. The width of the band is proportional to the grid size. We use finite-element spaces that are unfitted to the narrow band, so that elements are cut off. The implementation nevertheless is easy. We prove error estimates of optimal order for a Poisson equation on a surface and provide numerical tests and examples.

Keywords: elliptic equations; implicit surfaces; level sets; unfitted mesh finite-element method.

1. Introduction

Partial differential equations (PDEs) on surfaces occur in many applications. For example, traditionally they arise naturally in fluid dynamics and materials science and more recently in the mathematics of images. Denoting by Δ_Γ the Laplace–Beltrami operator on a hypersurface Γ contained in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, we consider the model problem

$$-\Delta_\Gamma u + cu = f \quad \text{on } \Gamma, \tag{1.1}$$

where c and f are prescribed data on the closed surface Γ. For strictly positive c, there is a unique solution (Aubin, 1982). A surface finite-element method was developed in Dziuk (1988) for approximating

---

†Corresponding author. Email: c.m.elliott@warwick.ac.uk

(1.1) using triangulated surfaces, see also Demlow & Dziuk (2007). This approach has been extended to parabolic equations (Dziuk & Elliott, 2007b) and the evolving surface finite-element method for parabolic conservation laws on time-dependent surfaces (Dziuk & Elliott, 2007a).

1.1 *Implicit surface equation*

In this paper we consider an Eulerian level set formulation of elliptic PDEs on an orientable hyper-surface Γ without boundary. The method is based on formulating the PDE on all level set surfaces of a prescribed function Φ whose zero level set is Γ and for which $\nabla\Phi$ does not vanish. Eulerian surface gradients are formulated by using a projection of the gradient in $\mathbb{R}^{n+1}$ onto the level surfaces of Φ. These Eulerian surface gradients are used to define weak forms of surface elliptic operators and so generate weak formulations of surface elliptic equations. The resulting equation is then solved in one dimension higher but can be solved on a finite-element mesh that is unaligned to the level sets of Φ. Using extensions of the data c and f, we solve the equation

$$-\Delta_\Phi u + cu = f \tag{1.2}$$

in all of Ω, where the Φ-Laplacian $\Delta_\Phi = \nabla_\Phi \cdot \nabla_\Phi$ is defined using the Φ-projected gradient

$$\nabla_\Phi u = P\nabla u, \quad P = I - \nu \otimes \nu, \quad \nu := \frac{\nabla\Phi}{|\nabla\Phi|}.$$

Equation (1.2) can be rewritten as

$$-\nabla \cdot (P\nabla u|\nabla\Phi|) + cu|\nabla\Phi| = f|\nabla\Phi| \quad \text{in } \Omega, \tag{1.3}$$

which can be viewed as a diffusion equation in an infinitesimally striated medium with an anisotropic diffusivity tensor $|\nabla\Phi|P$. This approach to surface PDEs consisting of solving implicitly on all level sets of a given function is due to Bertalmio *et al.* (2001) where finite-difference methods were proposed. The idea is to use regular Cartesian grids that are completely independent of the surface Γ. Finite-difference formulae for derivatives on the Cartesian grid are used to generate approximations of the implicit PDE.

1.2 *Finite-element approach*

It is natural to consider the finite-element approximation of implicit surface PDEs based on the following variational form of (1.3):

$$\int_\Omega \nabla_\Phi u \cdot \nabla_\Phi v |\nabla\Phi| + \int_\Omega cuv|\nabla\Phi| = \int_\Omega fv|\nabla\Phi|,$$

where v is an arbitrary test function in a suitable function space. This was proposed and developed in Burger (2008) where weighted Sobolev spaces were introduced. Then the existence and uniqueness of a solution to the variational formulation was proved by the Lax–Milgram lemma. Standard finite-element spaces can then be employed leading to a stable Galerkin algorithm.

1.3 *Issues, related works and contributions*

The finite-difference approximation of fourth-order parabolic equations on stationary implicit surfaces was considered in Greer *et al.* (2006). An interesting modification involving a different projection of the

surface was proposed by Greer (2006) and can be used to remove the degeneracy of the elliptic operator. Finite-difference methods for implicit equations on evolving surfaces were considered in Adalsteinsson & Sethian (2003) and Xu & Zhao (2003). A finite-element method for evolutionary second- and fourth-order implicit surface equations was formulated in Dziuk & Elliott (2008a). The evolving surface finite element method approach to conservation laws on evolving surfaces was extended to evolving implicit surfaces in Dziuk & Elliott (2008b). Recently, eigenvalue problems for elliptic implicit surface operators were considered in Brandman (2007).

With this implicit surface approach, the complexity associated with equations on manifolds embedded in a higher-dimensional space is removed at the expense of solving an equation in one space dimension higher. However, it is clear that the stability, efficiency and accuracy of the numerical methods will depend on the following factors.

1.3.1 *The degeneracy of the implicit equation and the regularity of solutions.* We observe that the anisotropic elliptic operator is degenerate in the sense that the diffusivity tensor P has a zero eigenvalue, $P\nu = 0$, and there is no diffusion in the normal direction. The solution on each surface only depends on the values of the data on that surface. Elliptic regularity is restricted to each surface, see Theorem 3.1. For example, the Φ-Laplace equation

$$-\Delta_\Phi u = 0$$

has the general solution, constant on each level surface of Φ,

$$u = g(\Phi),$$

where $g : \mathbb{R} \to \mathbb{R}$ is smooth, but weak solutions of the above PDE can be defined with g having no continuity properties.

Note that although the solution on a particular level set is independent of the solutions on any other (except by the extensions of the data) this will not be the case for the solution of the finite-element approximation where the discretization of the operator yields interdependence.

Another degeneracy for the equation is associated with critical points of Φ, where $\nabla \Phi = 0$. If Φ has an isolated critical point in Ω, then generically this is associated with the self-intersection of a level surface. It is easily observed that the finite-element approximation is still well defined but the analysis in the neighbourhood of such points is still open.

1.3.2 *Ω and the boundary conditions on $\partial\Omega$.* For an arbitrary domain Ω, the level sets of Φ will intersect the boundary $\partial\Omega$. It is then necessary to formulate boundary conditions for equation (1.2) in order to define a well-posed boundary-value problem. It is easy to see that there is a unique solution to the variational form (3.3) of the equation with the natural variational boundary condition

$$\nabla_\Phi u \cdot \nu_{\partial\Omega} = 0.$$

Note that, if Ω is a domain whose boundary is composed of level sets of Φ satisfying the condition

$$\Omega = \{x \in \mathbb{R}^{n+1} | \alpha < \Phi(x) < \beta\}, \tag{1.4}$$

with $-\infty < \alpha < \beta < \infty$, then this natural boundary condition is automatically satisfied.

1.3.3 *Well posedness and function spaces.* In order to formulate a well-posedness theory, we introduce implicit surface function spaces, see Section 2, $W_\Phi^{k,p}(\Omega)$ (e.g. $W_\Phi^{1,p}(\Omega) := \{f \in L^p(\Omega) | \nabla_\Phi f \in L^p(\Omega)\}$). In particular, we prove a density result for smooth functions in $W_\Phi^{1,p}(\Omega)$. Under condition (1.4), this density result allows us to establish the relation between $W_\Phi^{1,p}(\Omega)$ and the spaces $W^{1,p}(\Gamma_r)$ on the r-level sets $\Gamma_r = \{x \in \Omega | \Phi(x) = r\}$ of Φ. This leads to establishing the surface elliptic regularity result of Theorem 3.1,

$$\|u\|_{W_\Phi^{2,2}(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}.$$

Regularity across level sets is then established in Theorem 3.2 by assuming smoothness of the normal derivatives of the data.

1.3.4 *The choice of the computational domain.* It is possible to triangulate the domain Ω and define the finite-element approximation on this triangulated domain. However, here we wish to consider the approximation of the PDE on just one surface Γ, namely the zero level set of Φ. It is natural to consider defining the computational domain to be a narrow annular region containing Γ in the interior. A possible choice is the domain $\{x \in \Omega | |\Phi(x)| < \delta\}$. In principle, the finite-element approach has the attraction of being able to discretize arbitrary domains accurately using a union of triangles to define an approximating domain. However, we wish to avoid fitting the triangulation to the level surfaces of Φ in the neighbourhood of Γ. Our approach is based on the *unfitted finite-element method* (Barrett & Elliott, 1984, 1987, 1988), in which the domain on which we approximate the equation is not a union of elements. In this paper, we propose a narrow band finite-element method based on an unfitted mesh that is h-narrow in the sense that the domain of integration is $D_h := \{x \in \Omega | |\Phi_h(x)| < \gamma h\}$, where Φ_h is an interpolation of Φ. The finite-difference approach has to deal with the nonalignment of $\partial\Omega$ with the Cartesian grid and discretize the equation at grid points on the boundary of the computational domain. Previous works concerning finite-difference approximations have used Neumann and Dirichlet boundary conditions (Adalsteinsson & Sethian, 2003; Xu & Zhao, 2003; Greer, 2006; Greer *et al.*, 2006).

1.3.5 *Order of accuracy of the numerical solution.* Currently works on finite-difference approximations of implicit surface equations have focused on practical issues and there is no error analysis. The natural Galerkin error bound in a weighted energy norm was derived in Burger (2008). However, this is an error bound over all level surfaces contained in Ω. Since the finite-element mesh does not respect the surface it is not clear that the usual order of accuracy will hold on one level surface. The main theoretical numerical analysis result of this paper is an O(h) error bound in the $H^1(\Gamma)$-norm for our finite-element method based on piecewise linear finite elements on an unfitted h-narrow band.

1.4 *Outline of paper*

The organization of the paper is as follows. In Section 2 we introduce the notion of Φ-implicit surface derivatives and Φ-implicit surface function spaces $W_\Phi^{k,p}(\Omega)$. In particular, we prove a density result for smooth functions in $W_\Phi^{1,p}(\Omega)$ that allows us to establish the relation between the spaces $W^{1,p}(\Gamma_r)$ on the r-level sets of Φ and $W_\Phi^{1,p}(\Omega)$. The existence, uniqueness and regularity theory for the Φ-elliptic equation is addressed in Section 3. The finite-element approximation is described in Section 4 and error bounds are proved. Finally, in Section 5 the numerical implementation is discussed together with the results of numerical experiments.

2. Preliminaries

2.1 Φ-derivatives

In what follows we shall assume that the given hypersurface Γ is closed and can be described as the zero level set of a smooth function $\Phi : \bar{\Omega} \to \mathbb{R}$, i.e.

$$\Gamma = \{x \in \Omega | \Phi(x) = 0\} \quad \text{and} \quad \nabla\Phi(x) \neq 0, \ \forall x \in \bar{\Omega}. \tag{2.1}$$

Here Ω is a bounded domain in $\mathbb{R}^{n+1}$ that contains Γ. We define

$$\nu(x) := \frac{\nabla\Phi(x)}{|\nabla\Phi(x)|}, \quad x \in \Omega \tag{2.2}$$

as well as

$$P(x) := I - \nu(x) \otimes \nu(x), \quad x \in \Omega. \tag{2.3}$$

Note that P is an orthogonal projection satisfying $P = P^2 = P^{\mathrm{T}}$, where we use the superscript T to denote transpose. With a differentiable function $f : \Omega \to \mathbb{R}$, we associate its Φ-gradient

$$\nabla_\Phi f := \nabla f - (\nabla f \cdot \nu)\nu = P\nabla f.$$

Note that, restricting to any r-level set $\Gamma_r = \{x \in \Omega | \Phi(x) = r\}$ of Φ, $(\nabla_\Phi f)|_{\Gamma_r}$ is just the usual surface (tangential) gradient of $f|_{\Gamma_r}$ (cf. Gilbarg & Trudinger, 1983, Section 16.1). It is also the case that $\nu|_{\Gamma_r}$ is the unit normal to Γ_r pointing in the direction of increasing Φ. We define the mean curvature H by

$$H := -\sum_{i=1}^{n+1} \nu_{i,x_i}.$$

In what follows we shall sum from 1 to $n+1$ over repeated indices. Note that $H|_{\Gamma_r}$ is the mean curvature of the level surface Γ_r. We shall write

$$\nabla_\Phi f(x) = (\underline{D}_1 f(x), \dots, \underline{D}_{n+1} f(x)), \quad x \in \Omega$$

and have the following implicit surface integration formula (Dziuk & Elliott, 2008a):

$$\int_\Omega \nabla_\Phi f |\nabla\Phi| = -\int_\Omega f H \nu |\nabla\Phi| + \int_{\partial\Omega} f P \nu_{\partial\Omega} |\nabla\Phi|, \tag{2.4}$$

provided that f and $\partial\Omega$ are sufficiently smooth. Here $\nu_{\partial\Omega}$ is the unit outer normal to $\partial\Omega$.

LEMMA 2.1 Let $g : \Omega \to \mathbb{R}^{n+1}$ be a differentiable vector field with $g \cdot \nu = 0$ in Ω. Then

$$\nabla_\Phi \cdot g = \frac{1}{|\nabla\Phi|}\nabla \cdot (g|\nabla\Phi|) \quad \text{in } \Omega.$$

Proof. Recalling (2.2), we have

$$\frac{1}{|\nabla\Phi|}\nabla \cdot (g|\nabla\Phi|) = g_{i,x_i} + g_i \frac{\Phi_{x_j}\Phi_{x_jx_i}}{|\nabla\Phi|^2} = g_{i,x_i} + g_i \nu_j \frac{\Phi_{x_ix_j}}{|\nabla\Phi|}.$$

Differentiating the identity $g \cdot \nu = 0$ with respect to x_j, we obtain

$$g_{i,x_j} \nu_i = -g_i \nu_{i,x_j} = -g_i \frac{\Phi_{x_i x_j}}{|\nabla \Phi|}, \quad 1 \leqslant j \leqslant n+1,$$

which implies that

$$\frac{1}{|\nabla \Phi|} \nabla \cdot (g|\nabla \Phi|) = g_{i,x_i} - g_{i,x_j} \nu_i \nu_j = \nabla_\Phi \cdot g,$$

proving the desired result. $\qquad\square$

2.2 *Implicit surface function spaces*

Following Burger (2008), we introduce weak derivatives: for a function $f \in L^1_{\text{loc}}(\Omega), i \in \{1, \ldots, n+1\}$, we say that $g = \underline{D}_i f$ weakly if $g \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n+1})$ and

$$\int_\Omega f \underline{D}_i \zeta |\nabla \Phi| = -\int_\Omega g\zeta |\nabla \Phi| - \int_\Omega f\zeta H\nu_i |\nabla \Phi|, \quad \forall \zeta \in C_0^\infty(\Omega). \tag{2.5}$$

It is not difficult to see that (2.5) is equivalent to

$$\int_\Omega f \underline{D}_i \zeta = -\int_\Omega \underline{D}_i f \zeta + \int_\Omega f\zeta(h_i - H\nu_i), \quad \forall \zeta \in C_0^\infty(\Omega), \tag{2.6}$$

where $h_i = \nu_{i,x_j} \nu_j$ (see (3.8) below). Let $1 \leqslant p \leqslant \infty$. Then we define

$$W_\Phi^{1,p}(\Omega) := \{f \in L^p(\Omega) | \underline{D}_i f \in L^p(\Omega), \quad i = 1, \ldots, n+1\},$$

which we equip with the norm

$$\|f\|_{W_\Phi^{1,p}} := \left(\int_\Omega \left(|f|^p + \sum_{i=1}^{n+1} |\underline{D}_i f|^p \right) \right)^{\frac{1}{p}}, \quad p < \infty,$$

with the usual modification in the case $p = \infty$. In the same way, we can define the spaces $W_\Phi^{k,p}(\Omega)$ for $k \in \mathbb{N}, k \geqslant 2$. Also, let $W_\Phi^{0,p}(\Omega) = L^p(\Omega)$. We note that the spaces $W_\Phi^{k,2}(\Omega)$ are Hilbert spaces.

LEMMA 2.2 Let $1 \leqslant p < \infty$. Then $C^\infty(\Omega) \cap W_\Phi^{1,p}(\Omega)$ is dense in $W_\Phi^{1,p}(\Omega)$.

Proof. For $\epsilon > 0$, we set

$$u_\epsilon(x) := \int_\Omega \rho_\epsilon(x - y)u(y)\mathrm{d}y,$$

where ρ_ϵ is a standard mollifier. Employing a partition of unity, it is sufficient to prove that

$$\|u_\epsilon - u\|_{W_\Phi^{1,p}(\Omega')} \to 0, \quad \epsilon \to 0 \quad \text{for every } \Omega' \subset\subset \Omega.$$

Fix $\Omega' \subset\subset \Omega$ and choose Ω'' with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. For $0 < \epsilon < \text{dist}(\Omega', \partial\Omega'')$ and $x \in \Omega'$, we have

$$(\underline{D}_i u_\epsilon)(x) = \int_\Omega \rho_{\epsilon,x_i}(x-y)u(y)\mathrm{d}y - \int_\Omega \rho_{\epsilon,x_k}(x-y)\nu_k(x)\nu_i(x)u(y)\mathrm{d}y$$

$$= -\int_\Omega \underline{D}_i^y \rho_\epsilon(x-y)u(y)\mathrm{d}y - \int_\Omega \rho_{\epsilon,y_k}(x-y)r_{ik}(x,y)u(y)\mathrm{d}y,$$

where we have abbreviated:

$$r_{ik}(x,y) = \nu_k(y)\nu_i(y) - \nu_k(x)\nu_i(x).$$

Using (2.6), we obtain, for $0 < \delta < \text{dist}(\Omega'', \partial\Omega)$,

$$(\underline{D}_i u_\epsilon)(x) = \int_\Omega \rho_\epsilon(x-y)\underline{D}_i u(y)\mathrm{d}y - \int_\Omega \rho_\epsilon(x-y)u(y)(h_i(y) - H(y)\nu_i(y))\mathrm{d}y$$

$$- \int_\Omega \rho_{\epsilon,y_k}(x-y)r_{ik}(x,y)u_\delta(y)\mathrm{d}y - \int_\Omega \rho_{\epsilon,y_k}(x-y)r_{ik}(x,y)(u(y) - u_\delta(y))\mathrm{d}y.$$

Integrating by parts the third term and observing the relation

$$\frac{\partial}{\partial y_k}r_{ik}(x,y) = \frac{\partial}{\partial y_k}(\nu_k\nu_i)(y) = h_i(y) - H(y)\nu_i(y),$$

we may continue as follows:

$$(\underline{D}_i u_\epsilon)(x) = (\underline{D}_i u)_\epsilon(x) + \int_\Omega \rho_\epsilon(x-y)(u_\delta(y) - u(y))(h_i(y) - H(y)\nu_i(y))\mathrm{d}y$$

$$+ \int_\Omega \rho_\epsilon(x-y)r_{ik}(x,y)u_{\delta,y_k}\mathrm{d}y + \int_\Omega \rho_{\epsilon,y_k}(x-y)r_{ik}(x,y)(u_\delta(y) - u(y))\mathrm{d}y.$$

If we use standard arguments for convolutions and observe that $|r_{ik}(x,y)| \leqslant C|x-y|$ for $x \in \Omega'$ and $|x-y| \leqslant \epsilon$, we finally conclude

$$\|\underline{D}_i u_\epsilon - \underline{D}_i u\|_{L^p(\Omega')} \leqslant \|(\underline{D}_i u)_\epsilon - \underline{D}_i u\|_{L^p(\Omega')} + \|\underline{D}_i u_\epsilon - (\underline{D}_i u)_\epsilon\|_{L^p(\Omega')}$$

$$\leqslant \|(\underline{D}_i u)_\epsilon - \underline{D}_i u\|_{L^p(\Omega')} + C\|u_\delta - u\|_{L^p(\Omega'')} + C\frac{\epsilon}{\delta}\|u\|_{L^p(\Omega)}$$

$$\to 0,$$

as $\delta \to 0$ and $\frac{\epsilon}{\delta} \to 0$. $\square$

In what follows we assume in addition to (2.1) that Ω is of the form (1.4), i.e.

$$\Omega = \{x \in \mathbb{R}^{n+1} | \alpha < \Phi(x) < \beta\}, \quad -\infty < \alpha < \beta < \infty$$

and recall that $\Gamma_r = \{x \in \Omega | \Phi(x) = r\}$ is the r-level set of Φ. We suppose that Γ_α and Γ_β are closed hypersurfaces. The next result gives a characterization of functions in $W_\Phi^{1,p}(\Omega)$ in terms of the spaces $W^{1,p}(\Gamma_r)$ (see Aubin, 1982 for the corresponding definition).

COROLLARY 2.1 Let $1 < p < \infty$ and $u \in L^p(\Omega)$. Then

$$u \in W_\Phi^{1,p}(\Omega) \iff u_{|\Gamma_r} \in W^{1,p}(\Gamma_r) \text{ for almost all } r \in (\alpha, \beta), \text{ and } r \mapsto \|u\|_{W^{1,p}(\Gamma_r)} \in L^p(\alpha, \beta).$$

Proof. Let $u \in W_\Phi^{1,p}(\Omega)$. In view of Lemma 2.2, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\Omega) \cap W_\Phi^{1,p}(\Omega)$ such that $u_k \to u$ in $W_\Phi^{1,p}(\Omega)$. The coarea formula gives

$$\int_\alpha^\beta \int_{\Gamma_r} (|u_k - u|^p + |\nabla_\Phi(u_k - u)|^p)\mathrm{d}A\,\mathrm{d}r = \int_\Omega (|u_k - u|^p + |\nabla_\Phi(u_k - u)|^p)|\nabla\Phi| \to 0, \quad k \to \infty,$$

so that there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ with

$$\int_{\Gamma_r} (|u_{k_j} - u|^p + |\nabla_\Phi(u_{k_j} - u)|^p)\mathrm{d}A \to 0, \quad j \to \infty, \text{ for almost all } r \in (\alpha, \beta).$$

This implies that $u_{|\Gamma_r} \in W^{1,p}(\Gamma_r)$ for almost all $r \in (\alpha, \beta)$, while the fact that $r \mapsto \|u\|_{W^{1,p}(\Gamma_r)}$ belongs to $L^p(\alpha, \beta)$ is again a consequence of the coarea formula.

Assume now that $u|_{\Gamma_r} \in W^{1,p}(\Gamma_r)$ for almost all $r \in (\alpha, \beta)$, and $r \mapsto \|u\|_{W^{1,p}(\Gamma_r)} \in L^p(\alpha, \beta)$. Fix $i \in \{1, \ldots, n+1\}$. The coarea formula and integration by parts on Γ_r imply that, for $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega u\underline{D}_i\zeta|\nabla\Phi| = \int_\alpha^\beta \int_{\Gamma_r} u\underline{D}_i\zeta\,\mathrm{d}A\,\mathrm{d}r = -\int_\alpha^\beta \int_{\Gamma_r} (\nabla_{\Gamma_r}u)_i\zeta\,\mathrm{d}A\,\mathrm{d}r - \int_\alpha^\beta \int_{\Gamma_r} u\zeta H\nu_i\,\mathrm{d}A\,\mathrm{d}r.$$

Since $r \mapsto \|u\|_{W^{1,p}(\Gamma_r)}$ belongs to $L^p(\alpha, \beta)$, the linear functional

$$F_i(\zeta) := \int_\Omega u\underline{D}_i\zeta|\nabla\Phi| + \int_\Omega u\zeta H\nu_i|\nabla\Phi|$$

satisfies the estimate $|F_i(\zeta)| \leqslant C\|\zeta\|_{L^{p'}}$ for all $\zeta \in C_0^\infty(\Omega)$ (where $p'^{-1} = 1 - p^{-1}$). Since $p' < \infty$, $C_0^\infty(\Omega)$ is dense in $L^{p'}(\Omega)$ and hence there exists $v_i \in L^p(\Omega)$ satisfying $F_i(\zeta) = -\int_\Omega v_i\zeta|\nabla\Phi|$ for all $\zeta \in C_0^\infty(\Omega)$. Thus (2.5) holds and we infer that $u \in W_\Phi^{1,p}(\Omega)$. $\qquad\square$

3. Implicit surface elliptic equation

In this section we prove existence and regularity for solutions of the model equation (1.2).

3.1   *Φ-elliptic equation*

Let Ω and Φ be as above and assume that $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$ with

$$c \geqslant \bar{c} \quad \text{a.e. in } \Omega, \tag{3.1}$$

where $\bar{c}$ is a positive constant. We consider the implicit surface equation

$$-\Delta_\Phi u + cu = f \quad \text{in } \Omega.$$

Using Lemma 2.1, we may rewrite this equation in the form

$$-\nabla \cdot (P\nabla u|\nabla\Phi|) + cu|\nabla\Phi| = f|\nabla\Phi| \quad \text{in } \Omega. \tag{3.2}$$

Multiplying by a test function, integrating over Ω and using integration by parts leads to the following variational problem: find $u \in W_{\Phi}^{1,2}(\Omega)$ such that

$$\int_{\Omega} \nabla_{\Phi} u \cdot \nabla_{\Phi} v |\nabla \Phi| + \int_{\Omega} cuv |\nabla \Phi| = \int_{\Omega} fv |\nabla \Phi|, \quad \forall v \in W_{\Phi}^{1,2}(\Omega). \tag{3.3}$$

Note that the boundary term vanishes because the unit outer normal to $\partial \Omega$ points in the direction of $\nabla \Phi$. It follows from Burger (2008) that (3.3) has a unique solution $u \in W_{\Phi}^{1,2}(\Omega)$ and one can show that $u_{|\Gamma_r}$ is the weak solution of

$$-\Delta_{\Gamma} u + cu = f \quad \text{on } \Gamma_r$$

for almost all $r \in (\alpha, \beta)$. Since $f \in L^2(\Gamma_r)$ for almost all $r \in (\alpha, \beta)$, the regularity theory for elliptic PDEs on manifolds implies that $u \in W^{2,2}(\Gamma_r)$ for almost all $r \in (\alpha, \beta)$ and

$$\|u\|_{W^{2,2}(\Gamma_r)} \leqslant C \|f\|_{L^2(\Gamma_r)}.$$

Hence

$$\int_{\alpha}^{\beta} \|u\|_{W^{2,2}(\Gamma_r)}^2 \mathrm{d}r \leqslant C \int_{\alpha}^{\beta} \|f\|_{L^2(\Gamma_r)}^2 \mathrm{d}r = \int_{\Omega} |f|^2 |\nabla \Phi| < \infty,$$

so that Corollary 2.1 implies that $u \in W_{\Phi}^{2,2}(\Omega)$ with $\|u\|_{W_{\Phi}^{2,2}(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}$, and we have proved the following theorem.

THEOREM 3.1 Let $\Omega \subset \mathbb{R}^{n+1}$ satisfy (1.4) with $\Phi \in C^2(\overline{\Omega})$ and $\nabla \Phi \neq 0$ on $\overline{\Omega}$. Then there exists a unique solution $u \in W_{\Phi}^{2,2}(\Omega)$ of equation (3.3) and

$$\|u\|_{W_{\Phi}^{2,2}(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}. \tag{3.4}$$

3.2 *Regularity*

The goal of this section is to investigate the regularity of the solution of (3.3) in the normal direction. For this purpose, we define, for a differentiable function f,

$$D_{\nu} f := f_{x_i} \nu_i,$$

and similarly we say $z = D_{\nu} f$ weakly if

$$\int_{\Omega} f D_{\nu} \zeta = -\int_{\Omega} z \zeta + \int_{\Omega} f H \zeta \quad \forall \zeta \in C_0^{\infty}(\Omega).$$

The derivatives D_{ν} and $\underline{D}_i$ do not commute, but we have the following rules for exchanging the order of differentiation, which can be derived with the help of the symmetry relation $\underline{D}_i \nu_j = \underline{D}_j \nu_i$:

$$\underline{D}_i \underline{D}_j f - \underline{D}_j \underline{D}_i f = \alpha_{ij}^k \underline{D}_k f, \quad i, j = 1, \ldots, n+1, \tag{3.5}$$

$$\underline{D}_i D_{\nu} f - D_{\nu} \underline{D}_i f = h_i D_{\nu} f + \beta_i^k \underline{D}_k f, \quad i = 1, \ldots, n+1, \tag{3.6}$$

where

$$\alpha_{ij}^k = (\underline{D}_k v_j)v_i - (\underline{D}_k v_i)v_j, \tag{3.7}$$

$$h_i = v_{i,x_j} v_j = \frac{\underline{D}_i(|\nabla \Phi|)}{|\nabla \Phi|}, \tag{3.8}$$

$$\beta_i^k = v_i h_k + \underline{D}_i v_k. \tag{3.9}$$

Let us in the following assume that $\Phi \in C^3(\bar{\Omega})$. In order to examine the regularity of the solution of (3.3), we consider the regularized equation

$$-\epsilon D_v^2 u^\epsilon - \underline{D}_i \underline{D}_i u^\epsilon + c_\epsilon u^\epsilon = f_\epsilon \quad \text{in } \Omega, \tag{3.10}$$

$$\frac{\partial u^\epsilon}{\partial v_{\partial \Omega}} = 0 \quad \text{on } \partial \Omega, \tag{3.11}$$

in which f_ϵ and c_ϵ are mollifications of f and c with $c_\epsilon \geqslant \bar{c}$ in Ω. We associate with (3.10) and (3.11) the following weak form:

$$a_\epsilon(u^\epsilon, v) = \int_\Omega f_\epsilon v|, \nabla \Phi|, \quad \forall v \in H^1(\Omega), \tag{3.12}$$

where $a_\epsilon : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$a_\epsilon(u, v) := \epsilon \int_\Omega D_v u D_v v |\nabla \Phi| + \epsilon \int_\Omega b D_v u v |\nabla \Phi| + \int_\Omega \nabla_\Phi u \cdot \nabla_\Phi v |\nabla \Phi| + \int_\Omega c_\epsilon u v |\nabla \Phi|$$

and $b = (v_i |\nabla \Phi|)_{x_i} |\nabla \Phi|^{-1}$. It is not difficult to verify that the above problem is uniformly elliptic and that

$$a_\epsilon(u, u) \geqslant \frac{\epsilon}{2} \int_\Omega |D_v u|^2 |\nabla \Phi| + \int_\Omega |\nabla_\Phi u|^2 |\nabla \Phi| + \frac{\bar{c}}{2} \int_\Omega |u|^2 |\nabla \Phi| \tag{3.13}$$

for all $u \in H^1(\Omega)$ and $\epsilon \leqslant \epsilon_0$. In particular, (3.10) and (3.11) has a smooth solution u^ϵ that satisfies

$$\epsilon \int_\Omega |D_v u^\epsilon|^2 |\nabla \Phi| + \int_\Omega |\nabla_\Phi u^\epsilon|^2 |\nabla \Phi| + \int_\Omega |u^\epsilon|^2 |\nabla \Phi| \leqslant C \int_\Omega f_\epsilon^2 |\nabla \Phi| \leqslant C \int_\Omega f^2 |\nabla \Phi|. \tag{3.14}$$

Hence there exists a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ such that $\epsilon_k \searrow 0, k \to \infty$ and

$$u^{\epsilon_k} \rightharpoonup u \quad \text{in } W_\Phi^{1,2}(\Omega), \quad k \to \infty, \tag{3.15}$$

where u is the solution of (3.3). We shall examine the regularity of u in the normal direction by deriving bounds on the corresponding derivatives of u^ϵ that are independent of ϵ.

LEMMA 3.1 Let $\Omega \subset \mathbb{R}^{n+1}$ satisfy (1.4) with $\Phi \in C^2(\overline{\Omega})$ and $\nabla \Phi \neq 0$ on $\overline{\Omega}$. Suppose that $D_v c \in L^\infty(\Omega)$ and $D_v f \in L^2(\Omega)$. Then $D_v u \in W_\Phi^{1,2}(\Omega')$ for all $\Omega' \subset\subset \Omega$ and $v := \frac{1}{|\nabla \Phi|} D_v u$ is a weak solution of the equation

$$-\Delta_\Phi v + cv = g \quad \text{in } \Omega', \tag{3.16}$$

for all $\Omega' \subset\subset \Omega$, where

$$g = \frac{1}{|\nabla \Phi|} D_v f - \frac{1}{|\nabla \Phi|} u D_v c - \frac{1}{|\nabla \Phi|}(\underline{D}_i \beta_i^k \underline{D}_k u + (\beta_i^k + \beta_k^i)\underline{D}_i \underline{D}_k u - h_i \beta_i^k \underline{D}_k u). \tag{3.17}$$

Proof. Choose $r_1 < t_1 < t_2 < r_2$ and $\Omega'' \subset\subset \Omega$ such that

$$\Omega' \subset \{x \in \Omega | t_1 < \Phi(x) < t_2\} \subset \{x \in \Omega | r_1 < \Phi(x) < r_2\} \subset \Omega''$$

and let $\eta \in C_0^\infty(r_1, r_2)$ with $\eta = 1$ on $[t_1, t_2]$. Define $\zeta(x) := \eta(\Phi(x))$. Similarly to the proof of Lemma 2.2, it can be shown that

$$f_\epsilon, D_\nu f_\epsilon \to f, D_\nu f \quad \text{in } L^2(\Omega'') \tag{3.18}$$

$$c_\epsilon, D_\nu c_\epsilon \to c, D_\nu c \text{ a.e. in } \Omega'' \quad \text{and} \quad \|D_\nu c_\epsilon\|_{L^\infty} \leqslant C. \tag{3.19}$$

Applying D_ν to (3.10) and multiplying by $|\nabla \Phi|^{-1}$ yields

$$-\epsilon \frac{1}{|\nabla \Phi|} D_\nu^3 u^\epsilon - \frac{1}{|\nabla \Phi|} D_\nu \underline{D}_i \underline{D}_i u^\epsilon + c_\epsilon \frac{1}{|\nabla \Phi|} D_\nu u^\epsilon = \frac{1}{|\nabla \Phi|} D_\nu f_\epsilon - \frac{1}{|\nabla \Phi|} u^\epsilon D_\nu c_\epsilon \quad \text{in } \Omega. \tag{3.20}$$

Let us introduce $v^\epsilon := \frac{1}{|\nabla \Phi|} D_\nu u^\epsilon$. A straightforward calculation shows that

$$\frac{1}{|\nabla \Phi|} D_\nu^3 u^\epsilon = D_\nu^2 v^\epsilon + \gamma_1 D_\nu v^\epsilon + \gamma_2 v^\epsilon, \tag{3.21}$$

where

$$\gamma_1 = 2 \frac{1}{|\nabla \Phi|} D_\nu(|\nabla \Phi|) \quad \text{and} \quad \gamma_2 = \frac{1}{|\nabla \Phi|} D_\nu^2(|\nabla \Phi|).$$

Next, we deduce with the help of (3.6) that

$$\begin{aligned}
D_\nu \underline{D}_i \underline{D}_i u^\epsilon &= \underline{D}_i D_\nu \underline{D}_i u^\epsilon - h_i D_\nu \underline{D}_i u^\epsilon - \beta_i^k \underline{D}_k \underline{D}_i u^\epsilon \\
&= \underline{D}_i(\underline{D}_i D_\nu u^\epsilon - h_i D_\nu u^\epsilon - \beta_i^k \underline{D}_k u^\epsilon) - h_i \underline{D}_i D_\nu u^\epsilon + h_i h_i D_\nu u^\epsilon + h_i \beta_i^k \underline{D}_k u^\epsilon - \beta_i^k \underline{D}_k \underline{D}_i u^\epsilon \\
&= \underline{D}_i \underline{D}_i D_\nu u^\epsilon - \underline{D}_i(h_i D_\nu u^\epsilon) - \underline{D}_i \beta_i^k \underline{D}_k u^\epsilon - \beta_i^k \underline{D}_i \underline{D}_k u^\epsilon - h_i \underline{D}_i D_\nu u^\epsilon + h_i h_i D_\nu u^\epsilon \\
&\quad + h_i \beta_i^k \underline{D}_k u^\epsilon - \beta_i^k \underline{D}_k \underline{D}_i u^\epsilon. \tag{3.22}
\end{aligned}$$

On the other hand, (3.8) yields

$$\begin{aligned}
\underline{D}_i \underline{D}_i v^\epsilon &= \underline{D}_i \left(-h_i \frac{1}{|\nabla \Phi|} D_\nu u^\epsilon + \frac{1}{|\nabla \Phi|} \underline{D}_i D_\nu u^\epsilon \right) \\
&= \frac{1}{|\nabla \Phi|} (-\underline{D}_i(h_i D_\nu u^\epsilon) + h_i h_i D_\nu u^\epsilon + \underline{D}_i \underline{D}_i D_\nu u^\epsilon - h_i \underline{D}_i D_\nu u^\epsilon),
\end{aligned}$$

which combined with (3.22) implies that

$$\underline{D}_i \underline{D}_i v^\epsilon = \frac{1}{|\nabla \Phi|} (D_\nu \underline{D}_i \underline{D}_i u^\epsilon + \underline{D}_i \beta_i^k \underline{D}_k u^\epsilon + (\beta_i^k + \beta_k^i) \underline{D}_i \underline{D}_k u^\epsilon - h_i \beta_i^k \underline{D}_k u^\epsilon). \tag{3.23}$$

Inserting (3.21) and (3.23) into (3.20), we find that v^ϵ is a solution of the equation

$$-\epsilon D_\nu^2 v^\epsilon - \underline{D}_i \underline{D}_i v^\epsilon + c_\epsilon v^\epsilon = g_\epsilon, \tag{3.24}$$

where

$$g_\epsilon = \frac{1}{|\nabla\Phi|}D_\nu f_\epsilon - \frac{1}{|\nabla\Phi|}u^\epsilon D_\nu c_\epsilon + \epsilon\gamma_1 D_\nu v^\epsilon + \epsilon\gamma_2 v^\epsilon$$

$$- \frac{1}{|\nabla\Phi|}(\underline{D}_i\beta_i^k\underline{D}_k u^\epsilon + (\beta_i^k + \beta_k^i)\underline{D}_i\underline{D}_k u^\epsilon - h_i\beta_i^k\underline{D}_k u^\epsilon).$$

Multiplying (3.24) by $\zeta^2 v^\epsilon|\nabla\Phi|$, recalling that $\zeta = \eta(\Phi(x))$ and integrating over Ω, we obtain, in view of (2.5), the regularity of Φ and $\underline{D}_i\Phi = 0$, that

$$-\epsilon\int_\Omega \zeta^2 D_\nu^2 v^\epsilon v^\epsilon|\nabla\Phi| + \int_\Omega \zeta^2|\nabla_\Phi v^\epsilon|^2|\nabla\Phi| + \int_\Omega c_\epsilon\zeta^2|v^\epsilon|^2|\nabla\Phi| = \int_\Omega \zeta^2 g_\epsilon v^\epsilon|\nabla\Phi|. \qquad (3.25)$$

Integration by parts yields

$$-\epsilon\int_\Omega \zeta^2 D_\nu^2 v^\epsilon v^\epsilon|\nabla\Phi| = -\epsilon\int_\Omega \zeta^2(D_\nu v^\epsilon)_{x_i}v^\epsilon\nu_i|\nabla\Phi|$$

$$= \epsilon\int_\Omega \zeta^2|D_\nu v^\epsilon|^2|\nabla\Phi| + \epsilon\int_\Omega \zeta^2 D_\nu v^\epsilon v^\epsilon(\nu_i|\nabla\Phi|)_{x_i} + 2\epsilon\int_\Omega \zeta D_\nu\zeta D_\nu v^\epsilon v^\epsilon|\nabla\Phi|$$

$$\geqslant \frac{\epsilon}{2}\int_\Omega \zeta^2|D_\nu v^\epsilon|^2|\nabla\Phi| - C\epsilon\int_\Omega |v^\epsilon|^2|\nabla\Phi|. \qquad (3.26)$$

Let us next examine the terms on the right-hand side. Firstly,

$$\left|\int_\Omega \left(\frac{1}{|\nabla\Phi|}D_\nu f_\epsilon - \frac{1}{|\nabla\Phi|}u^\epsilon D_\nu c_\epsilon + \epsilon\gamma_1 D_\nu v^\epsilon + \epsilon\gamma_2 v^\epsilon\right)\zeta^2 v^\epsilon|\nabla\Phi|\right|$$

$$\leqslant \frac{\bar{c}}{4}\int_\Omega \zeta^2|v^\epsilon|^2|\nabla\Phi| + \frac{\epsilon}{4}\int_\Omega \zeta^2|D_\nu v^\epsilon|^2|\nabla\Phi| + C\int_{\Omega''}|D_\nu f_\epsilon|^2 + C\|D_\nu c_\epsilon\|_{L^\infty(\Omega'')}^2\int_\Omega |u^\epsilon|^2$$

provided that $0 < \epsilon \leqslant \epsilon_0$. Next

$$\left|\int_\Omega(\underline{D}_i\beta_i^k\underline{D}_k u^\epsilon - h_i\beta_i^k\underline{D}_k u^\epsilon)\zeta^2 v^\epsilon\right| \leqslant \frac{\bar{c}}{4}\int_\Omega \zeta^2|v^\epsilon|^2|\nabla\Phi| + C\int_\Omega \zeta^2|\nabla_\Phi u^\epsilon|^2|\nabla\Phi|.$$

Integration by parts gives

$$\int_\Omega \frac{1}{|\nabla\Phi|}(\beta_i^k + \beta_k^i)\underline{D}_i\underline{D}_k u^\epsilon\zeta^2 v^\epsilon|\nabla\Phi|$$

$$= -\int_\Omega \zeta^2\underline{D}_i\left(\frac{1}{|\nabla\Phi|}(\beta_i^k + \beta_k^i)\right)\underline{D}_k u^\epsilon v^\epsilon|\nabla\Phi| - \int_\Omega \frac{1}{|\nabla\Phi|}(\beta_i^k + \beta_k^i)\zeta^2\underline{D}_k u^\epsilon\underline{D}_i v^\epsilon|\nabla\Phi|$$

and hence

$$\left|\int_\Omega \frac{1}{|\nabla\Phi|}(\beta_i^k + \beta_k^i)\underline{D}_i\underline{D}_k u^\epsilon\zeta^2 v^\epsilon|\nabla\Phi|\right|$$

$$\leqslant \frac{\bar{c}}{4}\int_\Omega \zeta^2|v^\epsilon|^2|\nabla\Phi| + \frac{1}{2}\int_\Omega \zeta^2|\nabla_\Phi v^\epsilon|^2|\nabla\Phi| + C\int_\Omega |\nabla_\Phi u^\epsilon|^2|\nabla\Phi|.$$

Inserting the above estimates into (3.25) and recalling (3.14) as well as (3.18) and (3.19), we conclude that

$$\epsilon \int_{\Omega} \zeta^2 |D_\nu v^\epsilon|^2 |\nabla \Phi| + \int_{\Omega} \zeta^2 |\nabla_\Phi v^\epsilon|^2 |\nabla \Phi| + \int_{\Omega} \zeta^2 |v^\epsilon|^2 |\nabla \Phi|$$

$$\leqslant C \int_{\Omega''} |D_\nu f_\epsilon|^2 + C \int_{\Omega} (|u^\epsilon|^2 + |\nabla_\Phi u^\epsilon|^2) |\nabla \Phi| \leqslant C.$$

Since $\zeta \equiv 1$ on Ω', we infer that $(v^\epsilon)_{0 < \epsilon < \epsilon_0}$ is bounded in $W_\Phi^{1,2}(\Omega')$. Thus there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}, \epsilon_k \searrow 0$ such that (3.15) holds and

$$v^{\epsilon_k} \rightharpoonup v \in W_\Phi^{1,2}(\Omega'). \tag{3.27}$$

It is not difficult to show that this implies that $D_\nu u$ exists and satisfies $D_\nu u = |\nabla \Phi| v \in W_\Phi^{1,2}(\Omega')$. Let us finally derive the equation that is satisfied by v. For this purpose, we return to (3.24) and multiply this equation by $\varsigma |\nabla \Phi|$ for an arbitrary $\varsigma \in C_0^\infty(\Omega')$. Thus

$$-\epsilon_k \int_{\Omega'} D_\nu^2 v^{\epsilon_k} \varsigma |\nabla \Phi| + \int_{\Omega'} \underline{D}_i v^{\epsilon_k} \underline{D}_i \varsigma |\nabla \Phi| + \int_{\Omega'} c_{\epsilon_k} v^{\epsilon_k} \varsigma |\nabla \Phi| = \int_{\Omega'} g_{\epsilon_k} \varsigma |\nabla \Phi|.$$

Combining the relation $\epsilon_k D_\nu v^{\epsilon_k} \to 0$ in $L^2(\Omega')$ with (3.15) and (3.27) and the fact that $u \in W_\Phi^{2,2}(\Omega)$, one can pass to the limit in the above relation to obtain

$$\int_{\Omega'} \nabla_\Phi v \cdot \nabla_\Phi \varsigma |\nabla \Phi| + \int_{\Omega'} c v \varsigma |\nabla \Phi| = \int_{\Omega'} g \varsigma |\nabla \Phi|,$$

where g is given in (3.17). □

Observing that $g \in L^2(\Omega')$, we have proved the following regularity theorem.

THEOREM 3.2 In addition to the assumptions of Theorem 3.1, suppose that $\Phi \in C^3(\overline{\Omega})$ and that the coefficients satisfy $D_\nu c \in L^\infty(\Omega)$ and $D_\nu f \in L^2(\Omega)$. Then $D_\nu u \in W_\Phi^{2,2}(\Omega')$ for all $\Omega' \subset\subset \Omega$ and

$$\|D_\nu u\|_{W_\Phi^{2,2}(\Omega')} \leqslant C(\|f\|_{L^2(\Omega)} + \|D_\nu f\|_{L^2(\Omega)}). \tag{3.28}$$

The constant C depends on $\Omega, \Omega', \Phi, \bar{c}$ and c.

REMARK 3.1 Since $\nabla f = \nabla_\Phi f + D_\nu f \nu$, Theorems 3.1 and 3.2 imply that $u \in W^{1,2}(\Omega')$ for all $\Omega' \subset\subset \Omega$. Higher regularity of u can now be obtained by iterating the above arguments under appropriately strengthened hypotheses on the data of the problem. For example, estimating the second normal derivative would require deriving an equation for $w = |\nabla \Phi|^{-2} D_\nu^2 u$.

4. Numerical scheme and error analysis

4.1 *Finite-element approximation*

Let Ω_0 be an open polyhedral domain such that $B_\delta(\Gamma) \subset \Omega_0 \subset\subset \Omega$ for some $\delta > 0$, where $B_\delta(\Gamma) = \{x \in \Omega | \mathrm{dist}(x, \Gamma) < \delta\}$. Let $\mathcal{T}_h$ be a triangulation of Ω_0 with maximum mesh size $h := \max_{T \in \mathcal{T}_h} \mathrm{diam}(T)$. We suppose that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1} h$ and

contains a ball of radius κh. We denote by $a_1, \ldots, a_N$ the nodes of the triangulation and by $\psi_1, \ldots, \psi_N$ the corresponding linear basis functions, i.e. $\psi_i \in C^0(\bar{\Omega}_0)$ and $\psi_{i|T} \in P_1(T)$ for all $T \in \mathcal{T}_h$ satisfying $\psi_i(a_j) = \delta_{ij}$ for all $i, j = 1, \ldots, N$.

Let us define $\Phi_h := I_h \Phi = \sum_{i=1}^{N} \Phi(a_i)\psi_i$ as the usual Lagrange interpolation of Φ. In view of the smoothness of Φ, we have

$$\|\Phi - \Phi_h\|_{L^\infty(\Omega_0)} + h\|\nabla\Phi - \nabla\Phi_h\|_{L^\infty(\Omega_0)} \leqslant Ch^2, \tag{4.1}$$

from which we infer in particular that $|\nabla\Phi_h| \geqslant c_0$ in $\bar{\Omega}_0$ for $0 < h \leqslant h_0$. Hence we can define

$$P_h = I - \nu_h \otimes \nu_h, \quad \text{where } \nu_h = \frac{\nabla\Phi_h}{|\nabla\Phi_h|}.$$

Our aim is to estimate the error between u_h and the exact solution u. Since the equation for u is given in terms of the level set of Φ rather than that of Φ_h, it turns out to be useful to work with the transformation F_h defined in Lemma 4.1.

LEMMA 4.1 For h sufficiently small, there exists a bilipschitz mapping $F_h : \Omega_0 \to \Omega_0^h = F_h(\Omega_0)$ such that $\Phi \circ F_h = \Phi_h$ and

$$\|\text{id} - F_h\|_{L^\infty(\Omega_0)} + h\|I - DF_h\|_{L^\infty(\Omega_0)} \leqslant Ch^2. \tag{4.2}$$

Proof. We try to find F_h in the form

$$F_h(x) = x + \frac{\eta_h(x)}{|\nabla\Phi(x)|}\nu(x), \quad x \in \Omega_0,$$

where $\eta_h : \Omega_0 \to \mathbb{R}$ has to be determined. A Taylor expansion yields

$$\Phi(F_h(x)) = \Phi(x) + \nabla\Phi(x) \cdot \frac{\eta_h(x)}{|\nabla\Phi(x)|}\nu(x)$$

$$+ \frac{\eta_h(x)^2}{|\nabla\Phi(x)|^2}\int_0^1 (1-t)D^2\Phi\left(x + t\frac{\eta_h(x)}{|\nabla\Phi(x)|}\nu(x)\right)\,\mathrm{d}t\,\nu(x) \cdot \nu(x)$$

$$= \Phi(x) + \eta_h(x) + \frac{\eta_h(x)^2}{|\nabla\Phi(x)|^2}\int_0^1 (1-t)D^2\Phi\left(x + t\frac{\eta_h(x)}{|\nabla\Phi(x)|}\nu(x)\right)\,\mathrm{d}t\,\nu(x) \cdot \nu(x)$$

in view of (2.2). Hence $\Phi(F_h(x)) = \Phi_h(x)$ if and only if

$$\eta_h(x) = \Phi_h(x) - \Phi(x) - \frac{\eta_h(x)^2}{|\nabla\Phi(x)|^2}\int_0^1 (1-t)D^2\Phi\left(x + t\frac{\eta_h(x)}{|\nabla\Phi(x)|}\nu(x)\right)\,\mathrm{d}t\,\nu(x) \cdot \nu(x), \quad x \in \Omega_0. \tag{4.3}$$

This equation can be solved by applying Banach's fixed point theorem to the operator $S : B \to W^{1,\infty}(\Omega_0)$:

$$(S\psi)(x) := \Phi_h(x) - \Phi(x) - \frac{\psi(x)^2}{|\nabla\Phi(x)|^2}\int_0^1 (1-t)D^2\Phi\left(x + t\frac{\psi(x)}{|\nabla\Phi(x)|}\nu(x)\right)\,\mathrm{d}t\,\nu(x) \cdot \nu(x), \quad x \in \Omega_0.$$

Here B is the closed subset of $W^{1,\infty}(\Omega_0)$ given by

$$B := \{\psi \in W^{1,\infty}(\Omega_0)|\,\|\psi\|_{L^\infty(\Omega_0)} + h\|D\psi\|_{L^\infty(\Omega_0)} \leqslant Kh^2\},$$

where K is chosen sufficiently large and h is sufficiently small. We omit the details. $\qquad\square$

4.2 *Finite-element scheme*

Let us next describe the numerical scheme. The computational domain is taken as a narrow band of the form

$$D_h := \{x \in \Omega_0 \,|\, |\Phi_h(x)| < \gamma h\}.$$

Let $\mathcal{T}_h^C := \{T \in \mathcal{T}_h \,|\, T \cap D_h \neq \emptyset\}$ denote the computational elements and consider the nodes of all simplices T belonging to $\mathcal{T}_h^C$. After relabelling we may assume that these nodes are given by $a_1, \ldots, a_m$. Let $S_h := \mathrm{span}\{\psi_1, \ldots, \psi_m\}$. The discrete problem now reads: find $u_h \in S_h$ such that

$$\int_{D_h} P_h \nabla u_h \cdot \nabla v_h |\nabla \Phi_h| + \int_{D_h} c u_h v_h |\nabla \Phi_h| = \int_{D_h} f v_h |\nabla \Phi_h|, \quad \forall v_h \in S_h. \tag{4.4}$$

Our error analysis requires the conditions

$$\Gamma \subset \bigcup_{\Gamma \cap T \neq \emptyset} T \subseteq \bar{D}_h, \quad 0 < h \leqslant h_1, \tag{4.5}$$

and

$$|D_h| \leqslant Ch. \tag{4.6}$$

Condition (4.5) can be satisfied by choosing

$$\gamma = \max_{x \in \bar{\Omega}} |\nabla \Phi(x)|.$$

To see this, note that if $x \in \Gamma \cap T$ and $y \in T$ we have, because Φ_h is linear, that, for a vertex a of T,

$$|\Phi_h(y)| \leqslant |\Phi_h(a)| = |\Phi(a)| = |\Phi(a) - \Phi(x)| \leqslant h \max_{\bar{\Omega}} |\nabla \Phi| = \gamma h.$$

REMARK 4.1 In the case that Φ is a signed distance function then we can choose $\gamma = 1$.

The bound on the measure of D_h, (4.6), follows from the observation that, for h sufficiently small, $|\Phi| < 2\gamma h$ on D_h and, since $|\nabla \Phi| \geqslant c_\Phi > 0$ on Ω_0, using the coarea formula we have

$$|D_h| \leqslant \int_{-2\gamma h}^{2\gamma h} \int_{\Gamma_r} \frac{1}{|\nabla \Phi|} \leqslant \frac{4\gamma |\Gamma_*|}{c_\Phi} h,$$

where $|\Gamma_*|$ is a bound for the measure of the level sets of Φ in Ω_0.

4.3 *Error analysis*

THEOREM 4.1 Assume that the solution u of (3.3) belongs to $W^{2,\infty}(\Omega_0)$ and that $c, f \in W^{1,\infty}(\Omega_0)$. Let u_h be the solution of the finite-element scheme defined as in Section 4.2 satisfying (4.5). Then

$$\|u - u_h\|_{H^1(\Gamma)} \leqslant Ch.$$

Proof. We use the transformation F_h defined in Lemma 4.1. In order to derive the corresponding error relation, we take an arbitrary $v_h \in S_h$, multiply (3.2) by $v_h \circ F_h^{-1}$ and integrate over $D^h := F_h(D_h) =$

$\{x \in \Omega_0 | |\Phi(x)| < \gamma h\}$. Since the unit outer normal to ∂D^h points in the direction of $\nabla \Phi$, integration by parts yields

$$\int_{D^h} P \nabla u \cdot \nabla (v_h \circ F_h^{-1}) |\nabla \Phi| + \int_{D^h} c u v_h \circ F_h^{-1} |\nabla \Phi| = \int_{D^h} f v_h \circ F_h^{-1} |\nabla \Phi|, \quad \forall v_h \in S_h. \quad (4.7)$$

From $\Phi \circ F_h = \Phi_h$, it follows that

$$\nabla \Phi = (DF_h)^{-\mathrm{T}} \nabla \Phi_h,$$

which yields

$$P (DF_h)^{-\mathrm{T}} P_h = P \left((DF_h)^{-\mathrm{T}} - \frac{|\nabla \Phi|}{|\nabla \Phi_h|} \nu \nu_h^{\mathrm{T}} \right) = P (DF_h)^{-\mathrm{T}}.$$

Hence we have

$$P \nabla (v_h \circ F_h^{-1}) = P (DF_h)^{-\mathrm{T}} (P_h \nabla v_h) \circ F_h^{-1}. \quad (4.8)$$

Using the transformation rule, (4.8) and the fact that $P^2 = P$, we obtain

$$\int_{D^h} P \nabla u \cdot \nabla (v_h \circ F_h^{-1}) |\nabla \Phi|$$

$$= \int_{D_h} (P \nabla u) \circ F_h \cdot (P \nabla (v_h \circ F_h^{-1})) \circ F_h |(\nabla \Phi) \circ F_h| |\det DF_h|$$

$$= \int_{D_h} (P \nabla u) \circ F_h \cdot (DF_h)^{-\mathrm{T}} P_h \nabla v_h |(\nabla \Phi) \circ F_h| |\det DF_h|$$

$$= \int_{D_h} P_h \nabla u \cdot \nabla v_h |\nabla \Phi_h| + \int_{D_h} R_h \cdot P_h \nabla v_h,$$

where

$$|R_h| = |(DF_h)^{-1} (P \nabla u) \circ F_h |(\nabla \Phi) \circ F_h| |\det DF_h| - P_h \nabla u |\nabla \Phi_h||$$

$$\leqslant |[(DF_h)^{-1} |\det DF_h| - I] (P \nabla u) \circ F_h |(\nabla \Phi) \circ F_h||$$

$$\quad + |(P \nabla u) \circ F_h |(\nabla \Phi) \circ F_h| - P \nabla u |\nabla \Phi|| + |P \nabla u |\nabla \Phi| - P_h \nabla u |\nabla \Phi_h||$$

$$\leqslant C (\|DF_h - I\|_{L^\infty(\Omega_0)} + \|\nabla(\Phi - \Phi_h)\|_{L^\infty(\Omega_0)}) \|\nabla u\|_{L^\infty(\Omega_0)}$$

$$\quad + C (\|\nabla u\|_{W^{1,\infty}(\Omega_0)} + \|\nabla \Phi\|_{W^{1,\infty}(\Omega_0)}) \|F_h - \mathrm{id}\|_{L^\infty(\Omega_0)}.$$

Hence

$$\int_{D^h} P \nabla u \cdot \nabla (v_h \circ F_h^{-1}) |\nabla \Phi| = \int_{D_h} P_h \nabla u \cdot \nabla v_h |\nabla \Phi_h| + R_h^1(v_h),$$

while (4.1) and (4.2) imply that

$$|R_h^1(v_h)| \leqslant \left| \int_{D_h} R_h \cdot P_h \nabla v_h \right| \leqslant Ch|D_h|^{\frac{1}{2}} \|P_h \nabla v_h\|_{L^2(D_h)} \leqslant Ch^{\frac{3}{2}} \|P_h \nabla v_h\|_{L^2(D_h)}, \qquad (4.9)$$

where we have used the bound on $|D_h|$, (4.6). Since $c, f \in W^{1,\infty}(\Omega_0)$, one shows in a similar way that

$$\int_{D^h} c u v_h \circ F_h^{-1} |\nabla \Phi| = \int_{D_h} c u v_h |\nabla \Phi_h| + R_h^2(v_h),$$

$$\int_{D^h} f v_h \circ F_h^{-1} |\nabla \Phi| = \int_{D_h} f v_h |\nabla \Phi_h| + R_h^3(v_h),$$

where

$$|R_h^j(v_h)| \leqslant Ch^{\frac{3}{2}} \|v_h\|_{L^2(D_h)}, \quad j = 2, 3. \qquad (4.10)$$

Let us introduce $e : D_h \to \mathbb{R}$, $e := u - u_h$; in view of the above calculations, e satisfies the relation

$$\int_{D_h} P_h \nabla e \cdot \nabla v_h |\nabla \Phi_h| + \int_{D_h} c e v_h |\nabla \Phi_h| = R_h^3(v_h) - R_h^1(v_h) - R_h^2(v_h) \qquad (4.11)$$

for all $v_h \in S_h$. As usual, we split e as follows:

$$e = \left(u - \sum_{i=1}^m u(a_i) \psi_i \right) + \left(\sum_{i=1}^m u(a_i) \psi_i - u_h \right) \equiv \rho_h + \theta_h,$$

where ρ_h is the interpolation error and $\theta_h \in S_h$. Since $\cup_{T \in \mathcal{T}_h^C} T \subset B_{Ch}(\Gamma)$, we have

$$\|\rho_h\|_{H^1(D_h)}^2 \leqslant Ch \max_{T \in \mathcal{T}_h^C} \|\rho_h\|_{W^{1,\infty}(T)}^2 \leqslant Ch^3 \max_{T \in \mathcal{T}_h^C} \|D^2 u\|_{L^\infty(T)}^2 \leqslant Ch^3 \|D^2 u\|_{L^\infty(\Omega_0)}^2, \quad (4.12)$$

$$\|\rho_h\|_{H^1(\Gamma)} \leqslant C \|\rho_h\|_{W^{1,\infty}(\Gamma)} \leqslant Ch \|D^2 u\|_{L^\infty(\Omega_0)}. \qquad (4.13)$$

Next, inserting $v_h = \theta_h$ into (4.11) and recalling (4.9) and (4.10) as well as (4.12), we obtain

$$\int_{D_h} (|P_h \nabla \theta_h|^2 + |\theta_h|^2) \leqslant C \|\rho_h\|_{H^1(D_h)} \left(\int_{D_h} (|P_h \nabla \theta_h|^2 + |\theta_h|^2) \right)^{\frac{1}{2}} + C \sum_{j=1}^3 |R_h^j(\theta_h)|$$

$$\leqslant Ch^{\frac{3}{2}} \left(\int_{D_h} (|P_h \nabla \theta_h|^2 + |\theta_h|^2) \right)^{\frac{1}{2}}$$

and hence

$$\int_{D_h} (|P_h \nabla \theta_h|^2 + |\theta_h|^2) \leqslant Ch^3. \qquad (4.14)$$

Let us now turn to the error estimate on Γ. Using inverse estimates together with (4.1), (4.5) and (4.14), we infer

$$\|\theta_h\|_{H^1(\Gamma)}^2 = \int_{\Gamma} (|\theta_h|^2 + |P\nabla\theta_h|^2)$$

$$\leqslant \sum_{T \cap \Gamma \neq \emptyset} |T \cap \Gamma| (\|\theta_h\|_{L^\infty(T)}^2 + |(P_h\nabla\theta_h)_{|T}|^2 + Ch_T^2 |\nabla\theta_{h|T}|^2)$$

$$\leqslant C \sum_{T \cap \Gamma \neq \emptyset} h_T^n (h_T^{-(n+1)} \|\theta_h\|_{L^2(T)}^2 + |(P_h\nabla\theta_h)_{|T}|^2)$$

$$\leqslant C \sum_{T \cap \Gamma \neq \emptyset} h_T^{-1} \int_T (|\theta_h|^2 + |P_h\nabla\theta_h|^2)$$

$$\leqslant Ch^{-1} \int_{D_h} (|\theta_h|^2 + |P_h\nabla\theta_h|^2)$$

$$\leqslant Ch^2.$$

Combining this estimate with (4.13), we then obtain

$$\|u - u_h\|_{H^1(\Gamma)} \leqslant \|\rho_h\|_{H^1(\Gamma)} + \|\theta_h\|_{H^1(\Gamma)} \leqslant Ch$$

and the theorem is proved.   □

5. Numerical results

In this section we present the results of some numerical tests and experiments. In particular, they confirm the order h convergence in the $H^1(\Gamma)$-norm and suggest higher-order convergence in the $L^2(\Gamma)$-norm. A detailed description of the finite-element space was given in Section 4.2.

5.1 *Implementing the unfitted finite-element method*

As described in Section 4.2, we replace the smooth level set function Φ by its piecewise linear interpolant $\Phi_h = I_h\Phi$ and work on $D_h = \{x \in \Omega_0 | |\Phi_h(x)| < \gamma h\}$, yielding the finite-element scheme (4.4). The set D_h then consists of simplices T and sub-elements $\tilde{T}$ that are 'cut-off' simplices. In practice, it is not necessary to keep the mesh data structures for a uniformly refined triangulation of Ω_0 as elements not intersecting D_h play no part in the computation. So, in the implementation reported here, the domain Ω_0 is triangulated coarsely and then the grid is refined in a strip that includes the computational domain D_h.

For example, a typical situation for a curve in two space dimensions is shown in Fig. 1. The curve Γ intersects the two-dimensional grid quite arbitrarily, giving rise to sub-elements $\tilde{T}$ that are triangles or quadrilaterals of arbitrary size and regularity. The method then yields a standard linear system
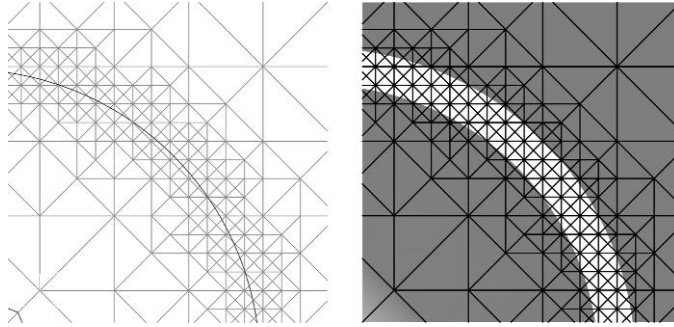
$$SU + MU = b,$$

FIG. 1. The curve Γ (red in the electronic version of the article) intersects the cartesian grid in an uncontrolled way (left). The unfitted finite-element method is applied to the white strip around the curve (right).

with stiffness matrix S, mass matrix M and right-hand side b given by

$$S_{ij} = \int_{D_h} P_h \nabla \psi_i \cdot \nabla \psi_j |\nabla \Phi_h|, \quad M_{ij} = \int_{D_h} c \psi_i \psi_j |\nabla \Phi_h|, \quad b_i = \int_{D_h} f \psi_i |\nabla \Phi_h|, \quad i, j = 1, \ldots, m,$$

where m denotes the number of vertices of the triangles belonging to $\mathcal{T}_h^C$. Note that this includes vertices lying outside D_h.

In the usual way, one assembles the matrices element by element. In the case of elements that intersect D_h, one has to calculate the sub-element mass matrix $M^{\tilde{T}}$ and the element stiffness matrix $S^{\tilde{T}}$:

$$M_{ij}^{\tilde{T}} = \int_{\tilde{T}} c \psi_i^T \psi_j^T |\nabla \Phi_h|, \quad S_{ij}^{\tilde{T}} = \int_{\tilde{T}} P_h \nabla \psi_i^T \cdot \nabla \psi_j^T |\nabla \Phi_h|, \quad i, j = 1, \ldots, m.$$

Here ψ_i^T denotes the ith local basis function on $T \supset \tilde{T}$. The quantity $\nabla \Phi_h$ is constant on $\tilde{T}$. For the computation of the integrals we split $\tilde{T}$ into simplices and use standard integration formulas on these.

Note that, since the sub-elements $\tilde{T}$ of the cut-off triangulation can be arbitrarily small, it is possible that the resulting equations for vertices in $\mathcal{T}_h^C$ that lie outside of D_h may have arbitrarily small elements. Thus, in this form, the resulting equations can have an arbitrarily large condition number. However, in our case of piecewise linear finite elements, by diagonal preconditioning the degenerate conditioning is removed. For higher-order finite elements this simple resolution of the ill-conditioning problem is not possible and for deeper insight into the stabilization of higher-order unfitted finite elements we refer to Heine (2008).

5.2 *Two space dimensions*

We begin with a test computation for a problem with a known solution so that we are able to calculate the error between the continuous and discrete solutions.

EXAMPLE 5.1 We are going to solve the PDE

$$-\Delta_\Gamma u + u = f \tag{5.1}$$

on the curve

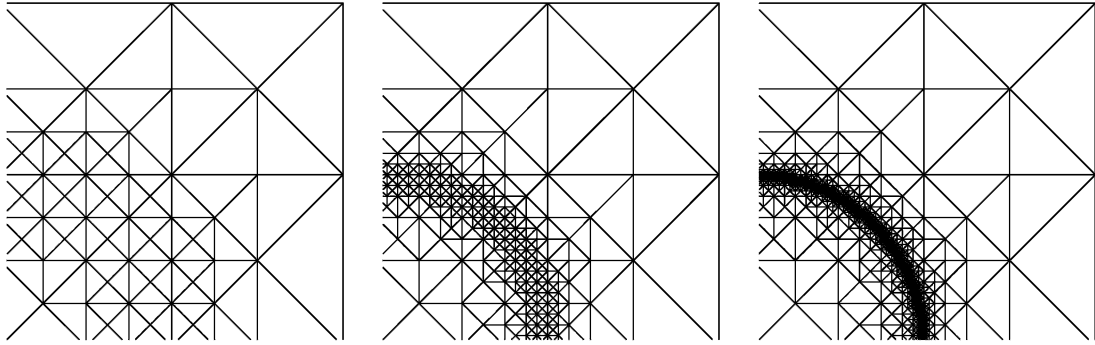$$\Gamma = \{x \in \mathbb{R}^2 | |x| = 1\}.$$

FIG. 2. Upper right part in $(0, 2) \times (0, 2)$ of the triangulation for Example 5.1. We show the triangulation levels 1 with 217 nodes, 3 with 997 nodes and 10 with 140677 nodes.

The level set function is chosen as $\Phi(x) = |x| - 1$. We choose $\Omega_0 = (-2, 2)^2$, which is triangulated as shown in Fig. 2. As the right-hand side for (5.1) we take

$$f(x) = 26(x_1^5 - 10x_1^3 x_2^2 + 5x_1 x_2^4), \quad x \in \Gamma,$$

and extend it constantly in the normal direction as

$$f(x) = \frac{26}{|x|^5}(x_1^5 - 10x_1^3 x_2^2 + 5x_1 x_2^4) = 26\cos(5\varphi), \quad x \in \Omega \setminus \{0\}.$$

The function

$$u(x) = \frac{1}{|x|^5}\frac{26|x|^2}{|x|^2 + 25}(x_1^5 - 10x_1^3 x_2^2 + 5x_1 x_2^4) = \frac{26r^2}{r^2 + 25}\cos(5\varphi) \quad (x = (r\cos\varphi, r\sin\varphi))$$

is then obviously the solution of the PDE (5.1) on the curve Γ, and a short calculation also shows that u, is a solution of

$$-\Delta_\Phi u + u = f, \quad x \in \mathbb{R}^2 \setminus \{0\}. \tag{5.2}$$

Figure 3 shows this solution and in Table 1 we give the errors and experimental orders of convergence (EOC) for this test problem. We calculated the errors on the strip $D_h = \{x \in \Omega_0 | |\Phi_h(x)| < \gamma h\}$:

$$E(L^2_\Phi(D_h)) = \left(\int_{D_h} (u - u_h)^2 |\nabla \Phi_h|\right)^{\frac{1}{2}} / |D_h|^{1/2},$$

$$E(H^1_\Phi(D_h)) = \left(\int_{D_h} |\nabla_{\Phi_h}(u - u_h)|^2 |\nabla \Phi_h|\right)^{\frac{1}{2}} / |D_h|^{1/2},$$

and on the curve $\Gamma_h = \{x \in \Omega_0 | \Phi_h(x) = 0\}$:

$$E(L^2(\Gamma_h)) = \|u - u_h\|_{L^2(\Gamma_h)}, \quad E(H^1(\Gamma_h)) = \|\nabla_{\Phi_h}(u - u_h)\|_{L^2(\Gamma_h)}.$$
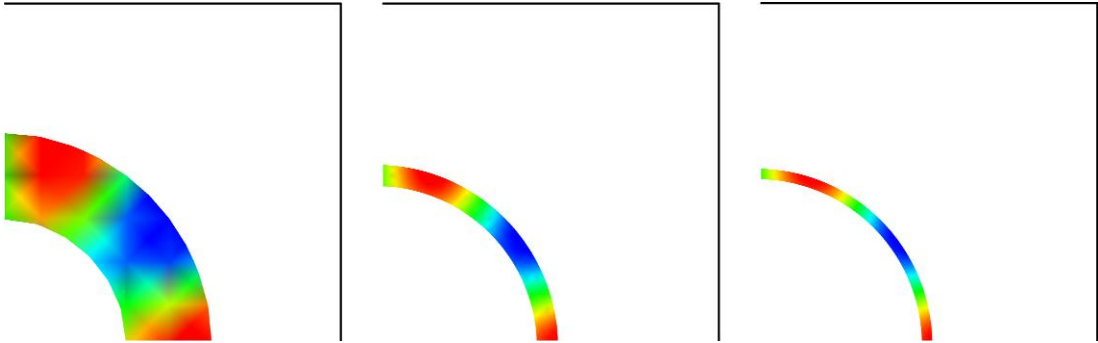
FIG. 3. Upper right part in $(0, 2) \times (0, 2)$ of the solution for Example 5.1. We show the levels 1, 3 and 4. The values of the solution are coloured linearly between -1 (blue) and 1 (red). The solution is shown in the strip $\{x \in (0, 2)^2 | ||x| - 1| \leqslant h\}$ only.

TABLE 1 *Errors and orders of convergence for Example* 5.1 *for the choice* $\gamma = 1$

h	$E(L_\Phi^2(D_h))$	EOC	$E(H_\Phi^1(D_h))$	EOC	$E(L^2(\Gamma_h))$	EOC	$E(H^1(\Gamma_h))$	EOC
0.5	0.1939	—	1.426	—	0.5744	—	4.382	—
0.25	0.05545	1.81	0.7293	0.97	0.1321	2.12	2.124	1.05
0.125	0.0191	1.54	0.3611	1.01	0.0424	1.64	1.089	0.96
0.0625	0.007185	1.41	0.1765	1.03	0.01339	1.66	0.4877	1.16
0.03125	0.002562	1.49	0.08708	1.02	0.004748	1.50	0.2548	0.94
0.01562	0.0008775	1.55	0.04319	1.01	0.001439	1.72	0.1234	1.05
0.007812	0.0003267	1.43	0.02153	1.00	0.0004022	1.84	0.06124	1.01
0.003906	0.0001366	1.26	0.01073	1.00	0.0001096	1.88	0.03013	1.02
0.001953	6.186×10^{-5}	1.14	0.005367	1.00	2.968×10^{-5}	1.88	0.01515	0.99
0.0009766	2.929×10^{-5}	1.08	0.002681	1.00	8.116×10^{-6}	1.87	0.00747	1.02
0.0004883	1.416×10^{-5}	1.05	0.00134	1.00	2.251×10^{-6}	1.85	0.003705	1.01

For errors $E(h_1)$ and $E(h_2)$ for the grid sizes h_1 and h_2, the experimental order of convergence is defined as

$$\text{EOC}(h_1, h_2) = \log \frac{E(h_1)}{E(h_2)} \left(\log \frac{h_1}{h_2} \right)^{-1}.$$

We also include a table for the choice $\gamma = 5$. The results in Tables 1 and 2 for the choices $\gamma = 1$ and $\gamma = 5$ confirm the theoretical results from Theorem 4.1 for the $H^1(\Gamma)$-norm.

Our analysis does not provide a higher-order convergence in the $L^2(\Gamma)$-norm. However, the numerical results indicate the possibility of quadratic convergence for the $L^2(\Gamma)$-norm in two space dimensions. However, this quadratic convergence may need h to be sufficiently small. Note that for the larger values of h the behaviour of the $L^2(\Gamma)$-norm in Table 2 is erratic, and for smaller values of h the $L^2(\Gamma)$-error is larger than that in the narrower band. We do not have an explanation for this. We observe that using the coarea formula as in the derivation of the bound (4.6) implies that $|D_h| \geqslant ch$, so that the estimate (4.14) suggests that the error in the weighted norm $E(H_\Phi^1(D_h))$ is of O(h). The numerical experiments agree with this.

We add a computation on an asymptotically larger strip. For this, we have chosen

$$D_h = \{x \in \Omega_0 | |\Phi_h(x)| < 2\sqrt{h}\}. \tag{5.3}$$

TABLE 2 *Errors and orders of convergence for Example* 5.1 *for the choice* $\gamma = 5$

h	$E(L_\Phi^2(D_h))$	EOC	$E(H_\Phi^1(D_h))$	EOC	$E(L^2(\Gamma_h))$	EOC	$E(H^1(\Gamma_h))$	EOC
0.125	0.01419		0.3679	—	0.03491	—	1.096	—
0.0625	0.00514	1.47	0.1855	0.99	0.008297	2.07	0.5249	1.06
0.03125	0.00245	1.07	0.09261	1.00	0.002219	1.90	0.2685	0.97
0.01562	0.0014	0.81	0.04605	1.01	0.001778	0.32	0.131	1.04
0.007812	0.0008785	0.67	0.02279	1.01	0.001899	−0.10	0.06599	0.99
0.003906	0.0004603	0.93	0.01121	1.02	0.001171	0.70	0.03174	1.06
0.001953	0.0001785	1.37	0.005517	1.02	0.0004752	1.30	0.01544	1.04
0.0009766	5.698×10^{-5}	1.65	0.002731	1.01	0.0001539	1.63	0.007531	1.04
0.0004883	1.645×10^{-5}	1.79	0.00136	1.01	4.405×10^{-5}	1.80	0.003714	1.02

TABLE 3 *Results for Example* 5.1 *for a computation on the larger strip* (5.3)

h	$E(L_\Phi^2(D_h))$	EOC	$E(H_\Phi^1(D_h))$	EOC	$E(L^2(\Gamma_h))$	EOC	$E(H^1(\Gamma_h))$	EOC
0.125	0.014		0.3665	—	0.03483	—	1.096	—
0.0625	0.004774	1.55	0.1852	0.99	0.008195	2.09	0.5251	1.06
0.03125	0.001929	1.31	0.09314	0.99	0.002028	2.01	0.2678	0.97
0.01562	0.0008406	1.20	0.04668	1.00	0.0004825	2.07	0.1299	1.04
0.007812	0.0003966	1.08	0.02337	1.00	0.0001211	1.99	0.06588	0.98
0.003906	0.0001933	1.04	0.01169	1.00	2.975×10^{-5}	2.02	0.03267	1.01
0.001953	9.558×10^{-5}	1.02	0.005848	1.00	7.343×10^{-6}	2.02	0.01619	1.01
0.0009766	4.741×10^{-5}	1.01	0.002925	1.00	1.838×10^{-6}	2.00	0.008052	1.01
0.0004883	2.363×10^{-5}	1.00	0.001463	1.00	4.747×10^{-7}	1.95	0.004004	1.01

The results for this case are shown in Table 3. Apparently, the $L^2(\Gamma)$-norm converges quadratically. The numerical analysis on Γ for larger strips remains an open question.

5.3 *Three space dimensions*

In three space dimensions we are solving PDEs on surfaces. The numerical method is principally the same as in two dimensions. But now we have to calculate mass and stiffness matrices on cut-off tetrahedra. As a test for the asymptotic errors we use a similar example to the previous two-dimensional example.

EXAMPLE 5.2 We choose $\Gamma = S^2$ and $\Omega_0 = (-2, 2)^3$ together with $\Phi(x) = |x| - 1$. For any constant a, the function

$$u(x) = a \frac{|x|^2}{12 + |x|^2}(3x_1^2 x_2 - x_2^3), \quad x \in \Omega_0 \setminus \{0\}$$

is a solution of (5.2) for the right-hand side

$$f(x) = a(3x_1^2 x_2 - x_2^3), \quad x \in \Omega_0 \setminus \{0\}.$$

For the computations, we have chosen $a = -\frac{13}{8}\sqrt{\frac{35}{\pi}}$. In Table 4 we show the errors and experimental orders of convergence for this example. They confirm our theoretical results and again indicate higher-order convergence in $L^2(\Gamma)$.

TABLE 4 *Three-dimensional results for Example 5.2 for the choice $\gamma = 1$*

h	$E(L_\Phi^2(D_h))$	EOC	$E(H_\Phi^1(D_h))$	EOC	$E(L^2(\Gamma_h))$	EOC	$E(H^1(\Gamma_h))$	EOC
0.866	0.0391	—	0.2598	—	0.2041	—	1.268	—
0.433	0.01339	1.55	0.1354	0.94	0.04091	2.32	0.5131	1.30
0.2165	0.004791	1.48	0.06889	0.97	0.01199	1.77	0.2677	0.94
0.1083	0.002024	1.24	0.03459	0.99	0.00409	1.55	0.1362	0.97
0.05413	0.0008095	1.32	0.01714	1.01	0.00157	1.38	0.06595	1.05
0.02706	0.0003299	1.30	0.008548	1.00	0.000484	1.70	0.03307	1.00
0.01353	0.0001471	1.16	0.004262	1.00	0.0001353	1.84	0.01638	1.01
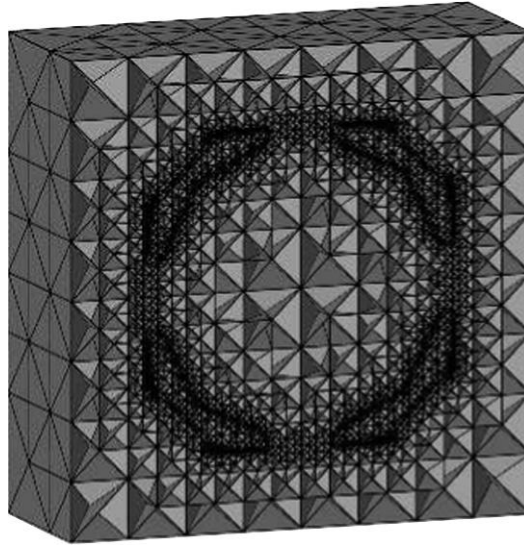
FIG. 4. Slice through the grid that was used for the computations from Example 5.3.

EXAMPLE 5.3 We end with a three-dimensional example. We solve the linear PDE (5.1) on a compli-cated two-dimensional surface $\Gamma = \{x \in \Omega_0 | \Phi(x) = 0\}$. The surface is given as the zero level set of the function

$$\Phi(x) = (x_1^2 + x_2^2 - 4)^2 + (x_3^2 - 1)^2 + (x_2^2 + x_3^2 - 4)^2 + (x_1^2 - 1)^2 + (x_3^2 + x_1^2 - 4)^2 + (x_2^2 - 1)^2 - 3. \quad (5.4)$$

The computational grid for this problem is shown in Fig. 4. As a right-hand side we have chosen the function

$$f(x) = 100 \sum_{j=1}^{4} \exp\left(-|x - x_{(j)}|^2\right), \quad (5.5)$$

with

$$x_{(1)} = (-1.0, 1.0, 2.04), \quad x_{(2)} = (1.0, 2.04, 1.0),$$
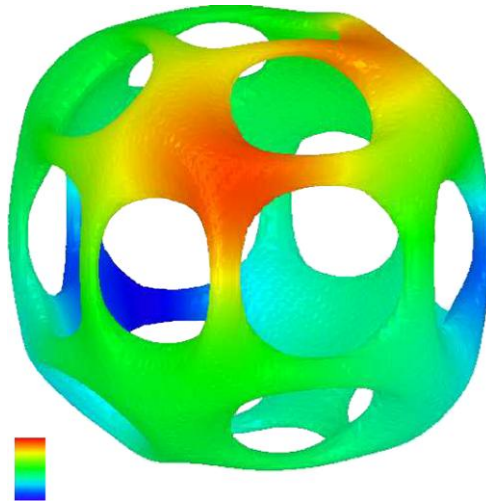$$x_{(3)} = (2.04, 0.0, 1.0), \quad x_{(4)} = (-0.5, -1.0, -2.04).$$

FIG. 5. Solution of the PDE (5.1) on the surface given by the zero level set of the function (5.4) for the right-hand side (5.5). The values of the solution are coloured according to the displayed scheme between the minimum −43.00 and the maximum 49.71.
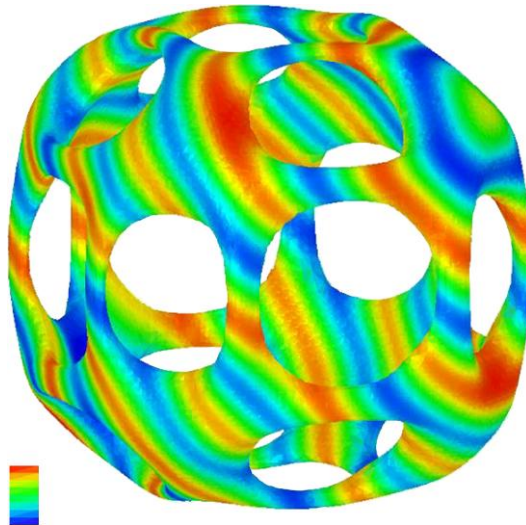

FIG. 6. Solution of the PDE (5.6) for the right-hand side (5.7). The surface is the same as in Fig. 5. Colouring ranges from the minimum −96.24 to the maximum 89.25 of the solution.

The points $x_{(j)}$ are close to the surface Γ. The triangulated domain is $\Omega_0 = (-3, 3)^3 \subset \mathbb{R}^3$. We used a three-dimensional grid with $m = 159880$ active nodes, i.e. nodes of the triangulation $\mathcal{T}_h^C$. The diameters of the simplices varied between 0.0001373291 and 0.03515625. In Fig. 5 we show the solution of this problem.

Figure 6 shows the solution of the PDE

$$-\Delta_\Gamma u + cu = f, \tag{5.6}$$

with $c = 100$ and the right-hand side

$$f(x) = 10000 \sin(5(x_1 + x_2 + x_3) + 2.5). \tag{5.7}$$

The computational data are the same as for Fig. 5.

REFERENCES

ADALSTEINSSON, D. & SETHIAN, J. A. (2003) Transport and diffusion of material quantities on propagating interfaces via level set methods. *J. Comput. Phys.*, **185**, 271–288.

AUBIN, TH. (1982) *Nonlinear Analysis on Manifolds. Monge–Ampère Equations*. Berlin: Springer.

BARRETT, J. W. & ELLIOTT, C. M. (1984) A finite element method for solving elliptic equations with Neumann data on a curved boundary using unfitted meshes. *IMA J. Numer. Anal.*, **4**, 309–325.

BARRETT, J. W. & ELLIOTT, C. M. (1987) Fitted and unfitted finite element methods for elliptic equations with smooth interfaces. *IMA J. Numer. Anal.*, **7**, 283–300.

BARRETT, J. W. & ELLIOTT, C. M. (1988) Finite element approximation of elliptic equations with Neumann or Robin condition on a curved boundary. *IMA J. Numer. Anal.*, **8**, 321–342.

BERTALMIO, M., CHENG, L. T., OSHER, S. & SAPIRO, G. (2001) Variational problems and partial differential equations on implicit surfaces. *J. Comput. Phys.*, **174**, 759–780.

BRANDMAN, J. (2007) A level-set method for computing the eigenvalues of elliptic operators defined on compact hypersurfaces. *J. Sci. Comput.*, **37**, 282–315.

BURGER, M. (2008) Finite element approximation of elliptic partial differential equations on implicit surfaces. *Comp. Vis. Sci.* (to appear). doi:10.1007/s00791-007-0081-x.

DEMLOW, A. & DZIUK, G. (2007) An adaptive finite element method for the Laplace–Beltrami operator on implicitly defined surfaces. *SIAM J. Numer. Anal.*, **45**, 421–442.

DZIUK, G. (1988) Finite elements for the Beltrami operator on arbitrary surfaces. *Partial Differential Equations and Calculus of Variations* (S. Hildebrandt & R. Leis eds). Lecture Notes in Mathematics, vol. 1357. Berlin: Springer, pp. 142–155.

DZIUK, G. & ELLIOTT, C. M. (2007a) Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27**, 262–292.

DZIUK, G. & ELLIOTT, C. M. (2007b) Surface finite elements for parabolic equations. *J. Comput. Math.*, **25**, 385–407.

DZIUK, G. & ELLIOTT, C. M. (2008a) Eulerian finite element method for parabolic equations on implicit surfaces. *Interfaces Free Bound.*, **10**, 119–138.

DZIUK, G. & ELLIOTT, C. M. (2008b) An Eulerian level set method for partial differential equations on evolving surfaces. *Comput. Vis. Sci.* (in press).

GILBARG, D. & TRUDINGER, N. S. (1983) *Elliptic Partial Differential Equations of Second Order*, 2nd edn. Grundlehren der Mathematischen Wissenschaften, vol. 224. Berlin: Springer.

GREER, J., BERTOZZI, A. & SAPIRO, G. (2006) Fourth order partial differential equations on general geometries. *J. Comput. Phys.*, **216**, 216–246.

GREER, J. B. (2006) An improvement of a recent Eulerian method for solving PDEs on general geometries. *J. Sci. Comput.*, **29**, 321–352.

HEINE, C.-J. (2008) Finite element methods on unfitted meshes. Preprint Fakultät für Mathematik und Physik, Universität Freiburg Nr. 08-09.

SCHMIDT, A. & SIEBERT, K. G. (2004) *Design of Aadaptive Finite Element Software. The Finite Element Toolbox ALBERTA*. Lecture Notes in Computational Science and Engineering, vol. 42. Berlin: Springer.

XU, J.-J. & ZHAO, H.-K. (2003) An Eulerian formulation for solving partial differential equations along a moving interface. *J. Sci. Comput.*, **19**, 573–594.