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A FULLY DISCRETE EVOLVING SURFACE FINITE ELEMENT METHOD*

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Abstract. In this paper we consider a time discrete evolving surface finite element method for the advection and diffusion of a conserved scalar quantity on a moving surface. In earlier papers using a suitable variational formulation in time dependent Sobolev space we proposed and analyzed a finite element method using surface finite elements on evolving triangulated surfaces [*IMA J. Numer Anal.*, 25 (2007), pp. 385–407; *Math. Comp.*, to appear]. Optimal order $L^2(\Gamma(t))$ and $H^1(\Gamma(t))$ error bounds were proved for linear finite elements. In this work we prove optimal order error bounds for a backward Euler time discretization.

Key words. surface finite elements, advection diffusion, moving surface, error analysis

AMS subject classifications. 65M60, 65M15, 35K99, 35R01, 35R37, 76R99

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1. Introduction. Partial differential equations on moving surfaces appear in a wide range of applications such as material science [3, 7], fluid flow [10, 12], and biology and biophysics [2, 8, 14, 16]. Numerical approaches include level set methods, surface finite elements, finite volume methods, and diffuse interface methods; see [1, 4, 5, 9, 13, 15, 17] and the references cited therein.

In this paper we focus on

$$\partial^\bullet u + u\,\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathcal{A}\nabla_\Gamma u) = 0 \tag{1.1}$$

on an evolving compact hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$, $m = 1, 2$, for time $t \in [0, T], T > 0$. The methods apply also to higher dimensional hypersurfaces. Note that although such a $\Gamma(t)$ does not have a boundary, the method may be extended to a hypersurface with a boundary on which an appropriate boundary condition is satisfied. Here $v_\mathcal{N}$ is the normal velocity of the surface, $v_\mathcal{T}$ is an advective tangential velocity field, $v = v_\mathcal{N} + v_\mathcal{T}$, ∇_Γ is the tangential surface gradient, $\mathcal{A}$ is a given diffusion tensor, and

$$\partial^\bullet u = u_t + v_\mathcal{N} \cdot \nabla u + v_\mathcal{T} \cdot \nabla_\Gamma u$$

is the material derivative.

This is an advection and diffusion equation (see [4] for a derivation) arising from conservation of a scalar field u on an evolving hypersurface with a flux q comprising a diffusive term and an advective tangential velocity v_τ so that

$$q = -\mathcal{A}\nabla_\Gamma u + v_\mathcal{T} \cdot u.$$

Note that the term $u\,\nabla_\Gamma \cdot v$ in (1.1) is needed for conservation and arises from the local shrinking or expansion of the surface.

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Our finite element method is based on the variational form (see [4])

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{A}\nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi, \tag{1.2}$$

where φ is an arbitrary test function defined on the space-time surface

$$\mathcal{G}_T = \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}.$$

A continuous in time finite element approximation was proposed and analyzed in [4] using piecewise linear finite elements on a triangulated surface interpolating $\Gamma(t)$ whose vertices move with the velocity v.

In this paper we prove optimal order error bounds for the following fully discrete finite element method:

$$\frac{1}{\tau}\left(\int_{\Gamma_h^{n+1}} U_h^{n+1}\phi_h^{n+1} - \int_{\Gamma_h^n} U_h^n\phi_h^n\right) + \int_{\Gamma_h^{n+1}} \mathcal{A}^{-l}\nabla_{\Gamma_h} U_h^{n+1}\cdot\nabla_{\Gamma_h}\phi_h^{n+1} = \int_{\Gamma_h^n} U_h^n \partial_h^\bullet \phi_h^n, \tag{1.3}$$

where U_h^n and ϕ_h^n belong to a piecewise linear finite element space, S_h^n, defined on a triangulation Γ_h^n of $\Gamma(t^n)$, $\tau > 0$ is the time step size, $\mathcal{A}^{-l}$ is an appropriate extension of the given diffusion tensor onto the discrete surface, and $\partial_h^\bullet \phi_h^n$ is a time discrete material derivative.

Using the transport property of the basis functions (see [4] and section 2 of this paper) this approximation can be written as

$$(\mathcal{M}^{n+1} + \tau\mathcal{S}^{n+1})\alpha^{n+1} = \mathcal{M}^n\alpha^n,$$

which we observe is a backward Euler discretization of the following system of ordinary differential equations:

$$\frac{d}{dt}\big(\mathcal{M}(t)\alpha\big) + \mathcal{S}(t)\alpha = 0,$$

arising from the continuous in time evolving surface finite element method (ESFEM) of [4], where $\mathcal{M}^n = \mathcal{M}(t^n)$ and $\mathcal{S}^n = \mathcal{S}(t^n)$, $\mathcal{M}(t)$ and $\mathcal{S}(t)$ are time dependent mass and stiffness matrices, and $\alpha^n = (\alpha_1^n, \ldots, \alpha_J^n)$ where $\{\alpha_j^n\}_{j=1}^J$ are the coefficients of the piecewise linear nodal basis functions $\{\chi_j^n\}_{j=1}^J$ such that

$$U_h^n = \sum_{j=1}^N \alpha_j^n \chi_j^n.$$

For the error between the continuous solution u and the continuous in time spatially discrete solution u_h (which is a lift onto $\Gamma(t)$ of the finite element solution on the triangulated surface) we proved in [4, 6] the bound

$$\sup_{t\in(0,T)} \|u(\cdot,t) - u_h(\cdot,t)\|^2_{L^2(\Gamma(t))} + h^2\int_0^T \|\nabla_\Gamma(u(\cdot,t) - u_h(\cdot,t))\|^2_{L^2(\Gamma(t))}dt \le ch^4,$$

where the constant c depends on norms of the continuous solution and the data of the problem. The purpose of this paper is to prove an optimal L^2-estimate for the error between the solutions of the continuous and fully discrete problems:

$$\|u(\cdot,t^n) - u_h^n\|^2_{L^2(\Gamma(t^n))} + h^2\tau\sum_{k=1}^n \|\nabla_\Gamma\big(u(\cdot,t^k) - u_h^k\big)\|^2_{L^2(\Gamma(t^k))} \le c(\tau^2 + h^4).$$

This is consistent with numerical experiments performed in [4, Table 2 for Example 7.3].

The challenges for convergence and stability analysis of discretization methods for advection-diffusion equations on moving surfaces arise because of the moving geometry. Existing results are for direct approximations based on interpolations of the evolving surface. An optimal order error analysis for the semidiscrete ESFEM based on piecewise linear finite element approximations has been given in [4, 6]. High order Runge–Kutta schemes for this semidiscretization have been derived and analyzed in [11]. In particular, stability and optimal order error bounds for the ordinary differential equation systems arising from ESFEM approximation of advection-diffusion equations on moving surfaces are obtained. In [13] Lenz, Nemadjieu, and Rumpf proved L^2 and H^1 error bounds for their proposed fully discrete time implicit finite volume scheme. The purpose of this paper is to extend the results of [5] to the case when the time discretization is based on a backward Euler scheme.

This paper is organized as follows. In section 2 we formulate the variational problem and the finite element method. In section 3 we derive discrete in time transport formulae and provide some estimates necessary for the error analysis. In section 4 we list some approximation properties established in [6]. Finally, in section 5 we prove the error bound.

2. The partial differential equation and finite element method.

2.1. Weak and variational formulations. In the following we assume that $\mathcal{A}$ is a sufficiently smooth symmetric $(m+1)\times(m+1)$ matrix which maps the tangent space of Γ at a point into itself and is positive definite on the tangent space, i.e.,

$$\mathcal{A}\xi\cdot\xi \ge c_0|\xi|^2 \quad \forall \xi\in\mathbb{R}^{m+1}, \quad \xi\cdot\nu = 0, \tag{2.1}$$

with some constant $c_0 > 0$. For the definition of a solution we assume that the elements of $\mathcal{A}$ belong to $L^\infty(\mathcal{G}_T)$.

For each $t\in[0,T]$ let $\Gamma(t)$ be a compact hypersurface oriented by the normal vector field $\nu(\cdot,t)$ and $\Gamma_0=\Gamma(0)$. We assume that there exists a map $G(\cdot,t):\Gamma_0\to\Gamma(t)$, $G\in C^1([0,T],C^2(\Gamma_0))$ such that $G(\cdot,t)$ is a diffeomorphism from Γ_0 to $\Gamma(t)$, and we set $v(G(\cdot,t),t)=G_t(\cdot,t)$, $G(\cdot,0)=Id$. We assume that $v(\cdot,t)\in C^2(\Gamma(t))$. The normal velocity of Γ then is defined by $v_{\mathcal{N}} = v\cdot\nu\nu$.

For $\varphi,\psi\in H^1(\Gamma)$ we define the bilinear forms

$$a(\varphi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \mathcal{A}(\cdot,t)\nabla_\Gamma\varphi(\cdot,t)\cdot\nabla_\Gamma\psi(\cdot,t), \tag{2.2}$$

$$m(\varphi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\psi(\cdot,t), \tag{2.3}$$

$$g(v(\cdot,t);\varphi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\psi(\cdot,t)\nabla_\Gamma\cdot v(\cdot,t). \tag{2.4}$$

Note that above we have suppressed the explicit dependence on t of the forms on the left-hand side. This dependence should be understood by the dependence on time of the arguments.

Using this notation, the variational form (1.2) becomes

$$\frac{d}{dt}m(u,\varphi) + a(u,\varphi) = m(u,\partial^\bullet\varphi). \tag{2.5}$$

In [4] we proved the existence of a weak solution. Furthermore for the initial data $u_0 \in H^1(\Gamma_0)$, $\Gamma_0 = \Gamma(0)$, and the elements of $\mathcal{A}$ and $v \in C^1(\mathcal{G}_T)$, the solution satisfies

$$\sup_{(0,T)} \|u\|^2_{L^2(\Gamma)} + \int_0^T \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} dt \le c\|u_0\|^2_{L^2(\Gamma_0)}, \tag{2.6}$$

$$\int_0^T \|\partial^\bullet u\|^2_{L^2(\Gamma)} dt + \sup_{(0,T)} \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} \le c\|u_0\|^2_{H^1(\Gamma_0)}, \tag{2.7}$$

where $c = c(\mathcal{A}, v, \mathcal{G}_T, T)$. Throughout the paper we use c to denote a generic constant which is independent of the approximation parameters h and τ but possibly dependent on the data.

2.2. Surface finite element method. Let N be a positive integer and set $\tau = T/N$. For each $n \in \{0, \dots, N\}$ set $t^n = n\tau$. For a discrete time sequence f^n, $n \in \{0, \dots, N\}$, we use the notation

$$\partial_\tau f^n = \frac{1}{\tau}\left(f^{n+1} - f^n\right).$$

2.2.1. Triangulated surface. The smooth evolving surface $\Gamma(t)$ ($\partial\Gamma(t) = \emptyset$) is approximated by an evolving surface

$$\Gamma_h(t) \subset \mathcal{N}(t),\ (\partial\Gamma_h(t) = \emptyset),$$

which for each t is of class $C^{0,1}$ and is smooth in time. We assume that $\Gamma_h(t)$ is homeomorphic to $\Gamma(t)$. Here $\mathcal{N}(t)$ is a neighborhood in $\mathbb{R}^{m+1}$ of $\Gamma(t)$ such that for each $x \in \mathcal{N}(t)$ there is a unique $p(x,t) \in \Gamma(t)$ which is the normal projection of x onto $\Gamma(t)$, i.e., $x - p(x,t) = d(x,t)\nu(p(x,t),t)$, where $|d(x,t)|$ is the distance of x to the hypersurface and $\nu(p(x,t),t)$ is normal to the surface. In particular for $m = 2$, $\Gamma_h(t)$ is a triangulated (and hence polyhedral) compact hypersurface consisting of simplices $E(t) \in \mathcal{T}_h(t)$, which form an admissible triangulation; i.e., each face of a simplex is always a face of an adjoining simplex. We suppose that the maximum diameter of the simplices in $\mathcal{T}_h(t)$ is bounded uniformly in time by h. Observe that this is an implicit restriction on the size of T for a given velocity v. Note that for each $E(t) \subset \Gamma_h(t)$ there is a unique $e(t) \subset \Gamma(t)$, $e(t) = p(E(t), t)$, whose edges are the unique projections of the edges of $E(t)$ onto $\Gamma(t)$. This induces an exact triangulation of $\Gamma(t)$ with curved edges.

In this paper we consider triangulated surfaces for which the vertices $\{X_j(t)\}_{j=1}^J$ of the simplices sit on $\Gamma(t)$ so that $\Gamma_h(t)$ is an interpolation. Furthermore we advect the nodes in the tangential direction with the advective velocity $v_\mathcal{T}$ and keep them on the surface using the normal velocity $v_\mathcal{N}$ so that

$$\frac{dX_j}{dt}(t) = v(X_j(t), t) \quad (j = 1, \dots, J). \tag{2.8}$$

2.2.2. Finite element spaces. We use piecewise linear finite elements on the evolving discrete surface $\Gamma_h(t)$ and $\mathcal{G}_T^h = \bigcup_{t\in[0,T]} \Gamma_h(t) \times \{t\}$. For any continuous functions η_h and η defined, respectively, on $\Gamma_h(t)$ and $\Gamma(t)$ we define extensions or lifts η_h^l and η^{-l} by extending constantly in the direction of the normal ν of the continuous surface so that $(\eta_h^l)^{-l} = \eta_h$; see [4, 6]. This has two purposes. First, we use the lift from $\Gamma_{h(t)}$ to $\Gamma(t)$ in order to define extensions of our finite element space which allows an error analysis of the discretization on the smooth surface $\Gamma(t)$.

Second, since the numerical method is based on integration of finite element functions on $\Gamma_h(t)$ we define approximations of the data, $\mathcal{A}$, given on $\Gamma(t)$ using $\mathcal{A}^{-l}$.

DEFINITION 2.1 (finite element spaces). *For each t and t^n we define the finite element spaces*

$$
\begin{aligned}
&S_h(t) = \left\{\phi_h(\cdot,t) \in C^0(\Gamma_h(t)) |\, \phi_h(\cdot,t)|_E \text{ is linear affine for each } E(t) \in \mathcal{T}_h(t)\right\},\\
&S_h^l(t) = \left\{\varphi_h(\cdot,t) = \phi_h(\cdot,t)^l |\, \phi_h(\cdot,t) \in S_h(t)\right\},\\
&S_h^n = S_h(t^n), \quad S_h^{n,l} = S_h^l(t^n).
\end{aligned}
$$

Remark 2.2. For each $\varphi_h \in S_h^l\big(\varphi_h^n \in S_h^{n,l}\big)$ there is a unique $\phi_h \in S_h\big(\phi_h^n \in S_h^n\big)$ such that $\varphi_h = \phi_h^l(\varphi_h^n = \phi_h^{n,l})$.

We denote by $\{\chi_j(\cdot,t)\}_{j=1}^J$ the piecewise linear basis functions from $S_h(t)$ such that $\chi_j(X_i(t),t) = \delta_{ij}$ and observe that $\{\chi_j^l(\cdot,t)\}_{j=1}^J$ form a basis of $S_h^l(t)$ and

$$
\begin{aligned}
\phi_h(\cdot,t) &= \sum_{j=1}^J \phi_j(t)\chi_j(\cdot,t) \in S_h(t),\\
\varphi_h(\cdot,t) = \phi_h^l(\cdot,t) &= \sum_{j=1}^J \phi_j(t)\chi_j^l(\cdot,t) \in S_h^l(t).
\end{aligned}
$$

Setting $\chi_j^n = \chi_j(\cdot,t^n), \chi_j^{n,l} = \chi_j^l(\cdot,t^n)$ for $j \in \{1,\ldots,J\}$, we use the notation

$$
\phi_h^n = \sum_{j=1}^J \phi_j^n \chi_j^n \in S_h^n, \quad \varphi_h^n = \phi_h^{n,l} = \sum_{j=1}^J \phi_j^n \chi_j^{n,l} \in S_h^{n,l}.
$$

We will find it convenient to define, for ϕ_h^n and ϕ_h^{n+1} in S_h^n and S_h^{n+1} for $t \in [t^n, t^{n+1}]$ and $\alpha = 0,1$,

$$
\underline{\phi}_h^{n+\alpha}(\cdot,t) = \sum_{j=1}^J \phi_j^{n+\alpha}\chi_j(\cdot,t) \in S_h(t), \tag{2.9}
$$

$$
\underline{\varphi}_h^{n+\alpha}(\cdot,t) = (\underline{\phi}_h^{n+\alpha}(\cdot,t))^l = \sum_{j=1}^J \phi_j^{n+\alpha}\chi_j^l(\cdot,t) \in S_h^l(t) \tag{2.10}
$$

as well as

$$
\begin{aligned}
\phi_h^L(\cdot,t) &= \frac{(t^{n+1}-t)}{\tau}\phi_h^n(\cdot) + \frac{(t-t^n)}{\tau}\phi_h^{n+1}(\cdot),\\
\varphi_h^L(\cdot,t) &= \frac{(t^{n+1}-t)}{\tau}\varphi_h^n(\cdot) + \frac{(t-t^n)}{\tau}\varphi_h^{n+1}(\cdot).
\end{aligned}
$$

2.2.3. Discrete material derivative. Associated with the moving triangulated surface $\Gamma_h(t)$ we define two material velocities V_h on Γ_h and v_h on Γ. First, for X_0 on $\Gamma_{h0} = \Gamma_h(0)$ we set $X(t)$ on the surface $\Gamma_h(t)$ by

$$
\dot{X}(t) = \sum_{j=1}^J \dot{X}_j(t)\chi_j(X(t),t), \quad X(0) = X_0,
$$

and define V_h to be the material velocity

$$V_h(x,t) = \sum_{j=1}^{J} \dot{X}_j(t)\chi_j(x,t), \quad x \in \Gamma_h(t). \tag{2.11}$$

It follows that V_h is an interpolation of v on $\Gamma_h(t)$ and we use it to define the finite element approximation. On the other hand, we wish to analyze the method on the smooth surface $\Gamma(t)$, and it is natural to construct a discrete material velocity on $\Gamma(t)$ which evolves the curvilinear triangles $e(t)$. For each $X(t)$ on $\Gamma_h(t)$ there is a unique $Y(t) = p(X(t),t) \in \Gamma(t)$,

$$\dot{Y}(t) = \frac{\partial p}{\partial t}(X(t),t) + V_h(X(t),t) \cdot \nabla p(X(t),t),$$

and we set

$$v_h(p(X(t),t),t) = \dot{Y}(t). \tag{2.12}$$

We observe the following:

– The edges of $e(t)$ are the projections onto $\Gamma(t)$ of the edges of $E(t) \subset \Gamma_h(t)$ and evolve with the material velocity $v_h(\cdot,t)$.

– The discrete material velocity v_h is *not* the interpolation of v in $S_h^l(t)$, which actually is given by

$$I_h v(p(x,t),t) = \sum_{j=1}^{J} \dot{X}_j(t)\chi_j^l(p(x,t),t) = \sum_{j=1}^{J} \dot{X}_j(t)\chi_j(x,t) = V_h(x,t)$$

for $x \in \Gamma_h(t)$.

Given the discrete velocity field $V_h \in (S_h)^{m+1}$ and the associated velocity field v_h on $\Gamma(t)$, (2.12), we define *discrete material derivatives* on $\Gamma_h(t)$ and $\Gamma(t)$ element by element through the equations

$$\partial_h^\bullet \phi_h|_{E(t)} = (\phi_{ht} + V_h \cdot \nabla \phi_h)|_{E(t)},$$
$$\partial_h^\bullet \varphi_h|_{e(t)} = (\varphi_{ht} + v_h \cdot \nabla \varphi_h)|_{e(t)}.$$

It was shown in [4, 6] that the basis functions satisfy the *transport property* that

$$\partial_h^\bullet \chi_j = 0, \quad \partial_h^\bullet \chi_j^l = 0, \tag{2.13}$$

and thus for $\phi_h \in S_h, \phi_j(t) = \phi_h(X_j(t),t)$,

$$\partial_h^\bullet \phi_h(\cdot,t) = \sum_{j=1}^{J} \dot{\phi}_j(t)\chi_j(\cdot,t) \in S_h(t), \tag{2.14}$$

and similarly for $\varphi_h = \phi_h^l \in S_h^l$,

$$\partial_h^\bullet \varphi_h = (\partial_h^\bullet \phi_h)^l = \sum_{j=1}^{J} \dot{\phi}_j \chi_j^l \in S_h^l. \tag{2.15}$$

We also have the estimate [6]

$$|\partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h| \le ch^2 |\nabla_\Gamma \varphi_h| \quad \text{on} \quad \Gamma. \tag{2.16}$$

From (2.15) we see that the material derivative and lifting process commute.

Definition 2.3 (time discrete material derivative). *Given $\phi_h^n \in S_h^n$ and $\phi_h^{n+1} \in S_h^{n+1}$ set*

$$\partial_h^\bullet \phi_h^n = \sum_{j=1}^J \partial_\tau \phi_j^n \chi_j^n \in S_h^n, \quad \partial_h^\bullet \varphi_h^n = \sum_{j=1}^J \partial_\tau \phi_j^n \chi_j^{n,l} \in S_h^{n,l}.$$

We observe that by definition the time discrete transport property of the basis functions hold:

$$\partial_h^\bullet \chi_j^n = 0, \quad \partial_h^\bullet \chi_j^{n,l} = 0,$$

and on $[t^n, t^{n+1}]$,

$$\partial_h^\bullet \underline{\phi}_h^{n+\alpha} = 0, \quad \partial_h^\bullet \underline{\varphi}_h^{n+\alpha} = 0$$

for $\alpha = 0, 1$. From the above we deduce

$$\underline{\phi}_h^{n+1}(\cdot, t^n) = \phi_h^n + \tau \partial_h^\bullet \phi_h^n, \quad \underline{\varphi}_h^{n+1}(\cdot, t^n) = \varphi_h^n + \tau \partial_h^\bullet \varphi_h^n. \tag{2.17}$$

2.2.4. Discrete bilinear forms. As discrete analogues of the bilinear forms (2.2), (2.3), and (2.4) we define for $\phi_h(\cdot, t), W_h(\cdot, t) \in S_h(t)$

$$a_h(\phi_h(\cdot,t), W_h(\cdot,t)) = \sum_{E(t)\in\mathcal{T}_h(t)} \int_{E(t)} \mathcal{A}^{-l}(\cdot,t) \nabla_{\Gamma_h} \phi_h(\cdot,t) \cdot \nabla_{\Gamma_h} W_h(\cdot,t), \tag{2.18}$$

$$m_h(\phi_h(\cdot,t), W_h(\cdot,t)) = \int_{\Gamma_h(t)} \phi_h(\cdot,t) W_h(\cdot,t), \tag{2.19}$$

$$g_h(V_h(\cdot,t); \phi_h(\cdot,t), W_h(\cdot,t)) = \int_{\Gamma_h(t)} \phi_h(\cdot,t) W_h(\cdot,t) \nabla_{\Gamma_h} \cdot V_h(\cdot,t). \tag{2.20}$$

We keep in mind that the forms explicitly depend on t too as discussed in section 2.1 for the time continuous forms. We indicate this in the time discrete case by writing for $\phi_h^n, W_h^n \in S_h^n$, $a_h(\phi_h^n, W_h^n)$, etc.

We will use the notation

$$|\phi_h|_{h,n} = \|\phi_h\|_{L^2(\Gamma_h(t^n))} = m_h(\phi_h^n, \phi_h^n)^{1/2}, \quad \phi_h \in S_h^n,$$

$$|\varphi|_n = \|\varphi\|_{L^2(\Gamma(t^n))} = m(\varphi^n, \varphi^n)^{1/2}, \quad \varphi \in H^1(\Gamma(t^n)).$$

2.3. Fully discrete scheme. We now can write the fully discrete scheme in the following form.

Given $U_h^0 \in S_h^0$ find $U_h^n \in S_h^n$, $n \in \{1, \dots, N\}$, such that for all $\phi_h^n \in S_h^n$ and $\phi_h^{n+1} \in S_h^{n+1}$ and $n \in \{0, \dots, N-1\}$,

$$\partial_\tau m_h(U_h^n, \phi_h^n) + a_h(U_h^{n+1}, \phi_h^{n+1}) = m_h(U_h^n, \partial_h^\bullet \phi_h^n). \tag{2.21}$$

Let us discuss the matrix-vector form of this fully discrete scheme. Choosing $\phi_h^n = \chi_i^n$ in (2.21), it follows that this definition is *equivalent* to

$$\partial_\tau m_h(U_h^n, \chi_i^n) + a_h(U_h^{n+1}, \chi_i^{n+1}) = 0 \tag{2.22}$$

for all $i = 1, \ldots, J$.

Setting $\mathcal{M}(t), \mathcal{M}^n$ to be the time dependent mass matrices

$$\mathcal{M}(t)_{jk} = \int_{\Gamma_h(t)} \chi_j(\cdot, t)\chi_k(\cdot, t), \quad \mathcal{M}^n = \mathcal{M}(t^n)$$

$(j, k = 1, \ldots, J)$ and $\mathcal{S}(t), \mathcal{S}^n$ to be the time dependent stiffness matrices

$$\mathcal{S}(t)_{jk} = \int_{\Gamma_h(t)} \mathcal{A}^{-l}(\cdot, t)\nabla_{\Gamma_h}\chi_j(\cdot, t) \cdot \nabla_{\Gamma_h}\chi_k(\cdot, t), \quad \mathcal{S}^n = \mathcal{S}(t^n),$$

we arrive at the following simple version of the fully discrete finite element approximation:

$$\partial_\tau \left(\mathcal{M}^n \alpha^n\right) + \mathcal{S}^{n+1}\alpha^{n+1} = 0.$$

Equivalently,

$$\left(\mathcal{M}^{n+1} + \tau\mathcal{S}^{n+1}\right)\alpha^{n+1} = \mathcal{M}^n\alpha^n, \tag{2.23}$$

where

$$U_h^{n+1} = \sum_{j=1}^{N} \alpha_j^{n+1}\chi_j^{n+1}$$

has to be determined as the solution of the sparse system of linear equations (2.23). Since for each n the mass matrix $\mathcal{M}^n$ is uniformly positive definite and the stiffness matrix $\mathcal{S}^n$ is positive semidefinite, we get existence and uniqueness of the discrete finite element solution.

2.4. Error bound. In section 4 we will prove the following error estimate for the fully discrete scheme.

THEOREM 2.4. *Let u be a sufficiently smooth solution of* (1.1) *and assume that*

$$\|u_0 - u_h^0\|_{L^2(\Gamma_0)} \le ch^2. \tag{2.24}$$

Let $u_h^k = \left(U_h^k\right)^l$ $(k = 0, \ldots, N)$ be the lift of the solution of the fully discrete scheme (2.21)*. Then for $0 < \tau \le \tau_0$ and $0 < h \le h_0$ we have the error bounds*

$$\|u(\cdot, t^n) - u_h^n\|^2_{L^2(\Gamma(t^n))} + h^2\tau\sum_{k=1}^{n} \|\nabla_\Gamma u(\cdot, t^k) - \nabla_\Gamma u_h^k\|^2_{L^2(\Gamma(t^k))} \le c(\tau^2 + h^4) \tag{2.25}$$

for all $n \in \{0, \ldots, N\}$ with a constant c which is independent of τ and h. Note that c, h_0, and τ_0 depend on the data of the problem.

3. Transport and Leibniz formulae.

3.1. Continuous in time transport and Leibniz formulae. The following formulae for the differentiation of time dependent surface integrals are called *transport formulae* and are proved in [4,6]. We use the notation $\partial^\bullet\mathcal{A}$ to denote the tensor which is obtained by taking the material derivative of $\mathcal{A}$ componentwise.

LEMMA 3.1 (transport theorems on surfaces). *Let $\mathcal{M}(t)$ be an evolving surface with normal velocity $v_\mathcal{N}$. Let $v_\mathcal{T}$ be a tangential velocity field on $\mathcal{M}(t)$. Let the* *boundary* $\partial\mathcal{M}(t)$ *evolve with the velocity* $v := v_{\mathcal{N}} + v_{\mathcal{T}}$. *Assume that* f *is a function such that all of the following quantities exist. Then*

$$\frac{d}{dt}\int_{\mathcal{M}(t)} f = \int_{\mathcal{M}(t)} \partial^\bullet f + f\,\nabla_\Gamma \cdot v. \tag{3.1}$$

We have for $\varphi, \psi \in H^1(\mathcal{G}_T)$,

$$\frac{d}{dt}m(\varphi,\psi) = m(\partial^\bullet\varphi,\psi) + m(\varphi,\partial^\bullet\psi) + g(v;\varphi,\psi) \tag{3.2}$$

and

$$\frac{d}{dt}a(\varphi,\psi) = a(\partial^\bullet\varphi,\psi) + a(\varphi,\partial^\bullet\psi) + b(v;\varphi,\psi) \tag{3.3}$$

with the bilinear form

$$b(v;\varphi,\psi) = \int_\Gamma \mathcal{B}(v)\nabla_\Gamma\varphi \cdot \nabla_\Gamma\psi. \tag{3.4}$$

With the deformation tensor $D(v)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1}\left(\mathcal{A}_{ik}\left(\nabla_\Gamma\right)_k v_j + \mathcal{A}_{jk}\left(\nabla_\Gamma\right)_k v_i\right)$ $(i,j = 1,\ldots,n+1)$ *and the tensor*

$$\mathcal{B}(v) = \partial^\bullet\mathcal{A} + \nabla_\Gamma \cdot v\,\mathcal{A} - 2D(v), \tag{3.5}$$

we have the formula

$$\frac{d}{dt}\int_{\mathcal{M}(t)} \mathcal{A}\nabla_\Gamma f\cdot\nabla_\Gamma g = \int_{\mathcal{M}(t)}\Big(\mathcal{A}\nabla_\Gamma\partial^\bullet f\cdot\nabla_\Gamma g + \mathcal{A}\nabla_\Gamma f\cdot\nabla_\Gamma\partial^\bullet g\Big) + \int_{\mathcal{M}(t)}\mathcal{B}(v)\nabla_\Gamma f\cdot\nabla_\Gamma g. \tag{3.6}$$

LEMMA 3.2 (transport theorems on triangulated surfaces). *Let* $\Gamma_h(t)$ *be an evolving admissible triangulation with material velocity* V_h. *Then*

$$\frac{d}{dt}\int_{\Gamma_h(t)} f = \int_{\Gamma_h(t)} \partial_h^\bullet f + f\,\nabla_{\Gamma_h}\cdot V_h. \tag{3.7}$$

For $\phi \in S_h(t), W_h \in S_h(t)$,

$$\frac{d}{dt}m_h(\phi,W_h) = m_h(\partial_h^\bullet\phi,W_h) + m_h(\phi,\partial_h^\bullet W_h) + g_h(V_h;\phi,W_h), \tag{3.8}$$

$$\frac{d}{dt}a_h(\phi,W_h) = a_h(\partial_h^\bullet\phi,W_h) + a_h(\phi,\partial_h^\bullet W_h) + b_h(V_h;\phi,W_h) \tag{3.9}$$

with the bilinear form

$$b_h(V_h;\phi,W_h) = \sum_{E(t)\in\mathcal{T}_h(t)}\int_{E(t)}\mathcal{B}_h(V_h)\nabla_{\Gamma_h}\phi\cdot\nabla_{\Gamma_h}W_h, \tag{3.10}$$

where

$$\mathcal{B}_h(V_h) = \partial_h^\bullet\mathcal{A}^{-l} + \nabla_{\Gamma_h}\cdot V_h\,\mathcal{A}^{-l} - 2D_h(V_h),$$

$$D_h(V_h)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1}\Big(\mathcal{A}_{ik}^{-l}\left(\nabla_\Gamma\right)_k V_{hj} + \mathcal{A}_{jk}^{-l}\left(\nabla_\Gamma\right)_k V_{hi}\Big), \quad i,j = 1,\ldots,n+1.$$

Let $\Gamma(t)$ be an evolving surface decomposed into curved elements $e(t)$ whose edges move with velocity v_h. Then

$$\frac{d}{dt}\int_{\Gamma(t)} f = \int_{\Gamma(t)} \partial_h^\bullet f + f\,\nabla_{\Gamma_h}\cdot v_h. \tag{3.11}$$

For $\varphi, w, \partial_h^\bullet\varphi, \partial_h^\bullet w \in H^1(\Gamma(t))$,

$$\frac{d}{dt}m(\varphi,w) = m(\partial_h^\bullet\varphi, w) + m(\varphi, \partial_h^\bullet w) + g(v_h;\varphi,w), \tag{3.12}$$

$$\frac{d}{dt}a(\varphi,w) = a(\partial_h^\bullet\varphi, w) + a(\varphi, \partial_h^\bullet w) + b(v_h;\varphi,w) \tag{3.13}$$

Remark 3.3. From the smoothness of Γ and $\mathcal{A}$ and the fact that V_h is the interpolant of the smooth velocity v we have that

$$\|\nabla_{\Gamma_h}V_h\|_{L^\infty(\Gamma_h)} + \|\mathcal{B}_h(V_h)\|_{L^\infty(\Gamma_h)} \le c$$

uniformly in time.

3.2. Discrete in time transport and Leibniz formulae. In this section we find an adequate form for discrete in time transport.

For $W_h(\cdot,t) \in S_h(t)$ and $w(\cdot,t) \in H^1(\Gamma(t))$, $(t \in [t^n, t^{n+1}])$ we set (as usual) $W_h^n = W_h(\cdot,t^n) \in S_h^n$ and $W_h^{n+1} = W_h(\cdot,t^{n+1}) \in S_h^{n+1}$. Similarly $w(\cdot,t^n) = w^n \in H^1(\Gamma(t^n))$ and $w(\cdot,t^{n+1}) = w^{n+1} \in H^1(\Gamma(t^{n+1}))$. For given $\phi_h^{n+1} \in S_h^{n+1}$ and its lifted version $\varphi_h^{n+1} = \phi_h^{n+1,l} \in S_h^{n+1,l}$ we define

$$\mathcal{L}_h(W_h,\phi_h^{n+1}) = \partial_\tau m_h(W_h^n,\phi_h^n) - m_h(W_h^n,\partial_h^\bullet\phi_h^n), \tag{3.14}$$

$$\mathcal{L}(w,\varphi_h^{n+1}) = \partial_\tau m(w^n,\varphi_h^n) - m(w^n,\partial_h^\bullet\varphi_h^n). \tag{3.15}$$

This is motivated by the fact that according to the fully discrete scheme (2.21) we will have to work with quantities of the form (3.14). We observe that this quantity depends only on W_h and on ϕ_h^{n+1} – and so the definition of $\mathcal{L}_h$ makes sense—since by Definition 2.3 we have

$$\begin{aligned}\mathcal{L}_h(W_h,\phi_h^{n+1}) &= \frac{1}{\tau}\left(m_h(W_h^{n+1},\phi_h^{n+1}) - m_h(W_h^n,\phi_h^n)\right) - \frac{1}{\tau}m_h\left(W_h^n, \sum_{j=1}^J(\phi_j^{n+1}-\phi_j^n)\chi_j^n\right)\\ &= \frac{1}{\tau}\left(m_h(W_h^{n+1},\phi_h^{n+1}) - m_h\left(W_h^n,\sum_{j=1}^J\phi_j^{n+1}\chi_j^n\right)\right).\end{aligned}$$

A similar argument holds for $\mathcal{L}$.

In the following lemma we will exploit the time continuous transport formula (3.2). For this we use the definitions (2.9) of $\underline{\phi}_h^{n+1}$ and (2.10) of $\underline{\varphi}_h^{n+1}$.

LEMMA 3.4. *Assume the situation defined above. Then*

$$\begin{aligned}&\mathcal{L}_h(W_h,\phi_h^{n+1})\\ &\quad = \frac{1}{\tau}\int_{t^n}^{t^{n+1}} m_h(\partial_h^\bullet W_h(\cdot,t),\underline{\phi}_h^{n+1}(\cdot,t)) + g_h(V_h(\cdot,t);W_h(\cdot,t),\underline{\phi}_h^{n+1}(\cdot,t))\,dt\end{aligned} \tag{3.16}$$

and
(3.17)

$$\mathcal{L}(w, \varphi_h^{n+1}) = \frac{1}{\tau}\int_{t^n}^{t^{n+1}} m(\partial_h^\bullet w(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t)) + g(v_h(\cdot,t); w(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t))\, dt.$$

Proof. Rewriting (3.14) we find that

$$\mathcal{L}_h(W_h, \phi_h^{n+1}) = \frac{1}{\tau}\big(m_h(W_h(\cdot,t^{n+1}), \underline{\phi}_h^{n+1}(\cdot,t^{n+1})) - m_h(W_h(\cdot,t^n), \underline{\phi}_h^{n+1}(\cdot,t^n))\big),$$

and using the transport formula (3.7) and the fact that $\partial_h^\bullet \underline{\phi}_h = 0$ we obtain the desired equation (3.16). Equation (3.17) follows by a similar calculation. ∎

LEMMA 3.5. *For $W_h(\cdot,t) \in S_h(t)$ and $w_h(\cdot,t) \in S_h^l(t)$, ($t \in [t^n, t^{n+1}]$), we have*

$$\partial_\tau m_h(W_h^n, W_h^n) - m_h(W_h^n, \partial_h^\bullet W_h^n) = \frac{1}{2}\partial_\tau m_h(W_h^n, W_h^n) + \frac{1}{2}\tau m_h(\partial_h^\bullet W_h^n, \partial_h^\bullet W_h^n)$$
$$+\frac{1}{2\tau}\left(m_h(W_h^{n+1}, W_h^{n+1}) - m_h(\underline{W}_h^{n+1}(\cdot,t^n), \underline{W}_h^{n+1}(\cdot,t^n))\right),$$
$$\partial_\tau m(w_h^n, w_h^n) - m(w_h^n, \partial_h^\bullet w_h^n) = \frac{1}{2}\partial_\tau m(w_h^n, w_h^n) + \frac{1}{2}\tau m(\partial_h^\bullet w_h^n, \partial_h^\bullet w_h^n)$$
$$+\frac{1}{2\tau}\left(m(w_h^{n+1}, w_h^{n+1}) - m(\underline{w}_h^{n+1}(\cdot,t^n), \underline{w}_h^{n+1}(\cdot,t^n))\right).$$

Proof. These identities follow by rearranging the terms and using the equations (2.17). ∎

3.3. Properties of the mass and stiffness matrices. For later use we have to control the time derivative of the mass and stiffness matrices and the related bilinear forms.

LEMMA 3.6. *For the continuous and the discrete bilinear forms we have the estimates*

$$|m_h(\phi_h^{n+1}, \phi_h^{n+1}) - m_h(\underline{\phi}_h^{n+1}(\cdot,t), \underline{\phi}_h^{n+1}(\cdot,t))| \le c\,\tau\, m_h(\phi_h^{n+1}, \phi_h^{n+1}), \tag{3.18}$$
$$|a_h(\phi_h^{n+1}, \phi_h^{n+1}) - a_h(\underline{\phi}_h^{n+1}(\cdot,t), \underline{\phi}_h^{n+1}(\cdot,t))| \le c\,\tau\, a_h(\phi_h^{n+1}, \phi_h^{n+1}), \tag{3.19}$$
$$|m(\varphi_h^{n+1}, \varphi_h^{n+1}) - m(\underline{\varphi}_h^{n+1}(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t))| \le c\,\tau\, m(\varphi_h^{n+1}, \varphi_h^{n+1}) \tag{3.20}$$

for $t \in [t^n, t^{n+1}]$ and

$$\big|m_h(W_h^{n+1}, W_h^{n+1}) - m_h(W_h^n + \tau\partial_h^\bullet W_h^n, W_h^n + \tau\partial_h^\bullet W_h^n)\big|$$
$$\le c\tau\, m_h(W_h^{n+1}, W_h^{n+1}), \tag{3.21}$$
$$\big|m(w_h^{n+1}, w_h^{n+1}) - m(w_h^n + \tau\partial_h^\bullet w_h^n, w_h^n + \tau\partial_h^\bullet w_h^n)\big| \le c\tau\, m(w_h^{n+1}, w_h^{n+1}) \tag{3.22}$$

if $\tau \le \tau_0$ with a time step size τ_0 which depends on the data of the problem.

We also have for $t \in [t^n, t^{n+1}]$

$$\|\underline{\phi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma_h(t))} \le c|\phi_h^{n+1}|_{n+1,h}, \|\underline{\varphi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma(t))} \le c|\varphi_h^{n+1}|_{n+1}, \tag{3.23}$$
$$\|\nabla_\Gamma \underline{\phi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma_h(t))} \le c|\nabla_\Gamma \phi_h^{n+1}|_{n+1,h}, \tag{3.24}$$
$$\|\nabla_\Gamma \underline{\varphi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma(t))} \le c|\nabla_\Gamma \varphi_h^{n+1}|_{n+1}.$$

Proof. Because of $\underline{\phi}_h^{n+1}(\cdot,t^{n+1}) = \phi_h^{n+1}$ and $\partial_h^\bullet \underline{\phi}_h^{n+1} = 0$ we have

$$\begin{aligned}
&m_h(\phi_h^{n+1},\phi_h^{n+1}) - m_h(\underline{\phi}_h^{n+1}(\cdot,t),\underline{\phi}_h^{n+1}(\cdot,t)) \\
&\quad = m_h(\underline{\phi}_h^{n+1}(\cdot,t^{n+1}),\underline{\phi}_h^{n+1}(\cdot,t^{n+1})) - m_h(\underline{\phi}_h^{n+1}(\cdot,t),\underline{\phi}_h^{n+1}(\cdot,t)) \\
&\quad = \int_t^{t^{n+1}} \frac{d}{ds} m_h(\underline{\phi}_h^{n+1}(\cdot,s),\underline{\phi}_h^{n+1}(\cdot,s))\,ds = \int_t^{t^{n+1}} \int_{\Gamma_h(s)} \nabla_{\Gamma_h}\cdot V_h(\cdot,s)\,\underline{\phi}_h^{n+1}(\cdot,s)^2\,ds \\
&\quad \geq -c\int_t^{t^{n+1}} m_h(\underline{\phi}_h^{n+1}(\cdot,s),\underline{\phi}_h^{n+1}(\cdot,s))\,ds.
\end{aligned}$$

For the nonnegative function $f(s) = m_h(\underline{\phi}_h^{n+1}(\cdot,s),\underline{\phi}_h^{n+1}(\cdot,s))$ this means

$$f(t^{n+1}) - f(t) \geq -c\int_t^{t^{n+1}} f(s)\,ds.$$

A standard Gronwall argument and the assumption $\tau \leq \tau_0$ lead to the estimate

$$f(t^{n+1}) - f(t) \geq -\tilde{c}\tau f(t^{n+1}).$$

This proves the estimate (3.18). The same argument holds for (3.20).

The estimates (3.21) and (3.22) now follow from (2.17) applied to $\phi_h = W_h$ and $\varphi_h = w_h$. Finally, we observe that a similar argument may be used to obtain (3.19). □

We add two estimates which will be used in the next section.

LEMMA 3.7. *Assume that* $\partial_h^\bullet \phi_h = 0$ *and* $\partial_h^\bullet \psi_h = 0$*. Then we have the estimates*

$$\begin{aligned}
&\left|m_h(\phi_h(\cdot,t^{k+1}),\psi_h(\cdot,t^{k+1}) - m_h(\phi_h(\cdot,t^k),\psi_h(\cdot,t^k)\right| \\
&\qquad \leq c\int_{t^k}^{t^{k+1}} m_h(\phi_h(\cdot,t),\phi_h(\cdot,t))^{\frac12} m_h(\psi_h(\cdot,t),\psi_h(\cdot,t))^{\frac12}\,dt, \qquad (3.25)\\
&\left|a_h(\phi_h(\cdot,t^{k+1}),\phi_h(\cdot,t^{k+1})) - a_h(\phi_h(\cdot,t^k),\phi_h(\cdot,t^k))\right| \\
&\qquad \leq c\int_{t^k}^{t^{k+1}} a_h(\phi_h(\cdot,t),\phi_h(\cdot,t))\,dt. \qquad (3.26)
\end{aligned}$$

Proof. We use the transport formula (3.7) and have

$$\begin{aligned}
&m_h(\phi_h(\cdot,t^{k+1}),\psi_h(\cdot,t^{k+1})) - m_h(\phi_h(\cdot,t^k),\psi_h(\cdot,t^k)) \\
&= \int_{t^k}^{t^{k+1}} g_h(V_h(\cdot,t);\phi_h(\cdot,t)\psi_h(\cdot,t))\,dt \leq c\int_{t^k}^{t^{k+1}} \|\phi_h(\cdot,t)\|_{L^2(\Gamma_h(t))}\|\psi_h(\cdot,t)\|_{L^2(\Gamma_h(t))}\,dt.
\end{aligned}$$

For the second estimate of the lemma we use the transport formula (3.9) and get

$$\begin{aligned}
&\left|a_h(\phi_h(\cdot,t^{k+1}),\phi_h(\cdot,t^{k+1})) - a_h(\phi_h(\cdot,t^k),\phi_h(\cdot,t^k))\right| \\
&\quad = \left|\int_{t^k}^{t^{k+1}} b_h(V_h(\cdot,t);\phi(\cdot,t)\phi(\cdot,t))\,dt\right| \leq c\int_{t^k}^{t^{k+1}} \|\nabla_{\Gamma_h(t)}\phi_h(\cdot,t)\|^2_{L^2(\Gamma_h)}\,dt \\
&\quad \leq c\int_{t^k}^{t^{k+1}} a_h(\phi_h(\cdot,t),\phi_h(\cdot,t))\,dt,
\end{aligned}$$

and the lemma is proved. □

4. Proof of the error bound.

4.1. Stability. We begin with a stability result which is the fully discrete analogue of the continuous estimates (2.6) and (2.7).

LEMMA 4.1. *The fully discrete solution U_h^k $(k = 0, \dots, N)$ satisfies the following a priori bounds for $\tau \le \tau_0$:*

$$|U_h^n|_{h,n}^2 + \tau c_0 \sum_{j=1}^{n} |\nabla_{\Gamma_h} U_h^k|_{h,k}^2 \le c|U_h^0|_{h,0}^2, \tag{4.1}$$

$$|u_h^n|_n^2 + \tau c_0 \sum_{j=1}^{n} |\nabla_{\Gamma} u_h^k|_k^2 \le c|u_h^0|_0^2, \tag{4.2}$$

$$\tau \sum_{k=0}^{n-1} |\partial_h^\bullet U_h^L(\cdot, t^{k+1} - 0)|_{h,k}^2 + c_0 |\nabla_{\Gamma_h} U_h^n|_{h,n}^2 \le c\big(|U_h^0|_{h,0}^2 + |\nabla_{\Gamma_h} U_h^0|_{h,0}^2\big), \tag{4.3}$$

$$\tau \sum_{k=0}^{n-1} |\partial_h^\bullet u_h^L(\cdot, t^{k+1} - 0)|_{k}^2 + c_0 |\nabla_{\Gamma} u_h^n|_{n}^2 \le c\big(|u_h^0|_{0}^2 + |\nabla_{\Gamma} u_h^0|_{0}^2\big).$$

The constants depend on the data of the problem including the final time T.

Proof. Using Lemma 3.5, the finite element equation (2.21), with $\phi_h = U_h$, can be written as

$$\frac{1}{2}\partial_\tau m_h(U_h^n, U_h^n) + \frac{1}{2}\tau m_h(\partial_h^\bullet U_h^n, \partial_h^\bullet U_h^n) + a_h(U_h^{n+1}, U_h^{n+1})$$
$$= -\frac{1}{2\tau}\left(m_h(U_h^{n+1}, U_h^{n+1}) - m_h(\underline{U}_h^{n+1}, \underline{U}_h^{n+1})\right).$$

We omit one positive term on the left-hand side and estimate the right-hand side with (3.18) to get

$$\frac{1}{2\tau}\left(|U_h^{n+1}|_{h,n+1}^2 - |U_h^n|_{h,n}^2\right) + c_0|\nabla_{\Gamma_h} U_h^{n+1}|_{h,n+1}^2 \le c|U_h^{n+1}|_{h,n+1}^2,$$

from which we obtain (4.1) by a discrete Gronwall argument.

The bound (4.2) follows by norm and seminorm equivalence for $|\phi_h^n|_{h,n}$ and $|\varphi_h^n|_n$ and $|\nabla_{\Gamma_h}\phi_h^n|_{h,n}$ and $|\nabla_\Gamma \varphi_h^n|_n$; see Lemma 5.2 in [4].

We derive the next bound using the matrix form of the discretization. Suppose that with vectors $w, v \in \mathbb{R}^J$ we have

$$\mathcal{M}^{n+1}w - \mathcal{M}^n v + \tau \mathcal{S}^{n+1} w = 0.$$

Then multiplying by $w - v$ and rearranging yields

$$\mathcal{M}^{n+1}(w-v)\cdot(w-v) + \frac{\tau}{2}\left(\mathcal{S}^{n+1}w\cdot w - \mathcal{S}^n v\cdot v\right) + \frac{\tau}{2}\mathcal{S}^{n+1}(w-v)\cdot(w-v)$$
$$= (\mathcal{M}^n - \mathcal{M}^{n+1})v\cdot(w-v) + \frac{\tau}{2}(\mathcal{S}^{n+1} - \mathcal{S}^n)v\cdot v.$$

We choose $v_j = U_j^k$ and $w_j = U_j^{k+1}$ $(j = 1, \dots, J)$ and find with $U^m = (U_1^m, \dots, U_J^m)$ for $m = k+1, k$ that

$$\mathcal{M}^{k+1}(U^{k+1} - U^k)\cdot(U^{k+1} - U^k) + \frac{\tau}{2}\left(\mathcal{S}^{k+1}U^{k+1}\cdot U^{k+1} - \mathcal{S}^k U^k \cdot U^k\right)$$
$$= -\frac{\tau}{2}\mathcal{S}^{k+1}(U^{k+1} - U^k)\cdot(U^{k+1} - U^k)$$
$$-(\mathcal{M}^{k+1} - \mathcal{M}^k)(U^{k+1} - U^k)\cdot U^k + \frac{\tau}{2}(\mathcal{S}^{k+1} - \mathcal{S}^k)U^k\cdot U^k.$$

If we rewrite this equation using the bilinear forms, then we arrive at

$$\begin{aligned}
&\tau^2 m_h(\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0),\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0)) + \frac{\tau}{2}\left(a_h(U_h^{k+1},U_h^{k+1}) - a_h(U_h^k,U_h^k)\right)\\
&\quad= -\frac{\tau^3}{2} a_h(\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0),\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0))\\
&\qquad -\tau\left(m_h(\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0),\underline{U}_h^k(\cdot,t^{k+1})) - m_h(\partial_h^\bullet U_h^L(\cdot,t^k+0),\underline{U}_h^k(\cdot,t^k))\right)\\
&\qquad +\frac{\tau}{2}\left(a_h(\underline{U}_h^k(\cdot,t^{k+1}),\underline{U}_h^k(\cdot,t^{k+1})) - a_h(\underline{U}_h^k(\cdot,t^k),\underline{U}_h^k(\cdot,t^k))\right).
\end{aligned}$$

After dropping the first term, using (3.25) and (3.26) together with (3.23) and (3.24), the right-hand side of this equation can be estimated from above by

$$\frac{1}{2}\tau^2 m_h(\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0),\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0)) + c\tau^2 m_h(U_h^k,U_h^k) + c\tau^2|\nabla_\Gamma U_h^{k+1}|_{h,k+1}^2.$$

Summation over k from 0 to $n-1$ and division by τ leads us to the estimate

$$\begin{aligned}
&\tau\sum_{k=0}^{n-1} m_h(\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0),\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0) + a_h(U_h^n,U_h^n) - a_h(U_h^0,U_h^0)\\
&\quad\le c\tau\sum_{k=0}^{n-1} m_h(U_h^k,U_h^k) + c\tau\sum_{k=0}^{n-1}|\nabla_{\Gamma_h}U_h^{k+1}|_{h,k+1}^2.
\end{aligned}$$

But this gives the estimate

$$\begin{aligned}
&\tau\sum_{k=0}^{n-1}|\partial_h^\bullet U_h^L(\cdot,t^{k+1}-0)|_{h,k+1}^2 + c_0|\nabla_{\Gamma_h}U_h^n|_{h,n}^2\\
&\quad\le c|\nabla_{\Gamma_h}U_h^0|_{h,0}^2 + c\tau\sum_{k=0}^{n-1}|\nabla_{\Gamma_h}U_h^{k+1}|_{h,k+1}^2 + c\tau\sum_{k=0}^{n-1}|U_h^k|_{h,k}^2.
\end{aligned}$$

We use the a priori estimate (4.1), and the estimate (4.3) is proved. ∎

4.2. Error decomposition and error equation. Setting $(\mathcal{R}_h u)(\cdot,t)$ to be the Ritz projection (see (A.6)) and $\rho(\cdot,t) = u(\cdot,t) - (\mathcal{R}_h u)(\cdot,t)$ it is convenient to introduce the error decomposition

$$u(\cdot,t^n) - u_h^n = \rho^n + \theta^n, \quad \rho^n = \rho(\cdot,t^n), \quad \theta^n = (\mathcal{R}_h u)(\cdot,t^n) - u_h^n \in S_h^{l,n}.$$

LEMMA 4.2. *The finite element solution satisfies the error relation*

$$\mathcal{L}(\theta,\varphi_h^{n+1}) + a(\theta^{n+1},\varphi_h^{n+1}) = F_2(\varphi_h^{n+1}) - F_1(\varphi_h^{n+1}) \quad \forall\varphi_h^{n+1}\in S_h^{n+1,l}, \tag{4.4}$$

where

$$\begin{aligned}
F_1(\varphi_h^{n+1}) &= \mathcal{L}(u_h,\varphi_h^{n+1}) - \mathcal{L}_h(U_h,\phi_h^{n+1})\\
&\quad + a(u_h^{n+1},\varphi_h^{n+1}) - a_h(U_h^{n+1},\phi_h^{n+1}),\\
F_2(\varphi_h^{n+1}) &= -\mathcal{L}(\rho,\varphi_h^{n+1}) + \mathcal{L}(u,\varphi_h^{n+1}) + a(u^{n+1},\varphi_h^{n+1}).
\end{aligned}$$

Proof. We begin by rewriting the finite element equation (2.21) on the time interval $[t^n,t^{n+1}]$ as

$$\mathcal{L}(u_h,\varphi_h^{n+1}) + a(u_h^{n+1},\varphi_h^{n+1}) = F_1(\varphi_h^{n+1})$$

On the other hand, using the definition (A.6) of the Ritz projection,

$$\mathcal{L}(\mathcal{R}_h u, \varphi_h^{n+1}) + a(\mathcal{R}_h u^{n+1}, \varphi_h^{n+1}) = F_2(\varphi_h^{n+1}) \quad \forall \varphi_h \in S_h^l.$$

Taking the difference of these two equations gives the desired result. ∎

Equation (4.4) is the basis of the error bound. We proceed by bounding $F_1(\varphi_h^{n+1})$ and $F_2(\varphi_h^{n+1})$.

LEMMA 4.3. *For every $\epsilon > 0$ we have the estimate*

$$|F_1(\varphi_h^{n+1})| \leq c(\epsilon)\frac{h^4}{\tau}\int_{t^n}^{t^{n+1}} \|\partial_h^\bullet u_h^L\|_{L^2(\Gamma)}^2 + \|u_h^L\|_{H^1(\Gamma)}^2\,dt + c(\epsilon)h^4|\nabla_\Gamma u_h^{n+1}|_{n+1}^2 + \epsilon|\nabla_\Gamma \varphi_h^{n+1}|_{n+1}^2 + c|\varphi_h^{n+1}|_{n+1}^2.$$

Proof. It is convenient to write

$$F_1(\varphi_h^{n+1}) = \big(\mathcal{L}(u_h, \phi_h^{n+1}) - \mathcal{L}_h(U_h, \phi_h^{n+1})\big) + \big(a(u_h^{n+1}, \varphi_h^{n+1}) - a_h(U_h^{n+1}, \phi_h^{n+1})\big) = I + II.$$

We observe that $\mathcal{L}_h(U_h, \phi_h^{n+1}) = \mathcal{L}_h(U_h^L, \phi_h^{n+1})$, $\mathcal{L}(u_h, \varphi_h^{n+1}) = \mathcal{L}(u_h^L, \varphi_h^{n+1})$, and $u_h^L = (U_h^L)^l$. Using (3.17) and (3.16) and the estimates (A.1) and (A.3) we find

$$\begin{aligned}|I| &\leq c\frac{h^2}{\tau}\int_{t^n}^{t^{n+1}} \|\partial_h^\bullet u_h^L(\cdot,t)\|_{L^2(\Gamma)}\|\underline{\varphi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma)} + \|u_h^L\|_{H^1(\Gamma)}\|\underline{\varphi}_h^{n+1}\|_{H^1(\Gamma)}\,dt \\ &\leq \varepsilon|\nabla_\Gamma \varphi_h^{n+1}|_{n+1}^2 + c(\varepsilon)\frac{h^4}{\tau}\int_{t^n}^{t^{n+1}} \|\partial_h^\bullet u_h^L\|_{L^2(\Gamma)}^2 + \|u_h^L\|_{H^1(\Gamma)}^2\,dt + c|\varphi_h^{n+1}|_{n+1}^2,\end{aligned}$$

where we have used (3.23) and (3.24). By (A.2) we have that

$$|II| \leq ch^2|\nabla_\Gamma u_h^{n+1}|_{n+1}|\nabla_\Gamma \varphi_h^{n+1}|_{n+1} \leq \varepsilon|\nabla_\Gamma \varphi_h^{n+1}|_{n+1}^2 + c(\varepsilon)h^4|\nabla_\Gamma u_h^{n+1}|_{n+1}^2.$$

The result is proved. ∎

LEMMA 4.4. *For F_2 we have the estimate*

$$\begin{aligned}|F_2(\varphi_h^{n+1})| &\leq c\frac{h^4}{\tau}\int_{t^n}^{t^{n+1}} \|u\|_{H^2(\Gamma)}^2 + \|\partial^\bullet u\|_{H^2(\Gamma)}^2\,dt \\ &+c(\epsilon)\tau\int_{t^n}^{t^{n+1}} \|u\|_{H^1(\Gamma)}^2 + \|\partial^\bullet u\|_{H^1(\Gamma)}^2\,dt + \epsilon|\nabla\varphi_h^{n+1}|_{n+1}^2 + c|\varphi_h^{n+1}|_{n+1}^2\end{aligned}$$

for arbitrary $\epsilon > 0$.

Proof. It follows from the variational form (2.5) that

$$m(u(\cdot,t^{n+1}), \underline{\varphi}_h^{n+1}(\cdot,t^{n+1})) - m(u(\cdot,t^n), \underline{\varphi}_h^{n+1}(\cdot,t^n)) + \int_{t^n}^{t^{n+1}} a(u(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t))\,dt = \int_{t^n}^{t^{n+1}} m(u(\cdot,t), \partial^\bullet \underline{\varphi}_h^{n+1}(\cdot,t))\,dt.$$

Since $\underline{\varphi}_h^{n+1}(\cdot,t^{n+1}) = \varphi_h^{n+1}$ and $\underline{\varphi}_h^{n+1}(\cdot,t^n) = \varphi_h^n + \tau\partial_h^\bullet\varphi_h^n$ we find that

$$\begin{aligned}&m(u(\cdot,t^{n+1}), \varphi_h^{n+1}) - m(u(\cdot,t^n), \varphi_h^n) + \int_{t^n}^{t^{n+1}} a(u(\cdot,t), \underline{\varphi}_h^{n+1}(\cdot,t))\,dt \\ &\quad = \tau m(u(\cdot,t^n), \partial_h^\bullet\varphi_h^n) + \int_{t^n}^{t^{n+1}} m(u(\cdot,t), \partial^\bullet \underline{\varphi}_h^{n+1}(\cdot,t))\,dt.\end{aligned}$$

Hence

$$\mathcal{L}(u,\varphi_h^{n+1})+\frac{1}{\tau}\int_{t^n}^{t^{n+1}} a(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t))dt=\frac{1}{\tau}\int_{t^n}^{t^{n+1}} m(u(\cdot,t),\partial^\bullet\underline{\varphi}_h^{n+1}(\cdot,t))\,dt,$$

from which we deduce that

$$F_2(\varphi_h^{n+1})=-\mathcal{L}(\rho,\varphi_h^{n+1})+\frac{1}{\tau}\int_{t^n}^{t^{n+1}} a(u(\cdot,t^{n+1}),\varphi_h^{n+1})-a(u(\cdot,t),\underline{\varphi}_h^{n+1}(\cdot,t))\,dt$$
$$+\frac{1}{\tau}\int_{t^n}^{t^{n+1}} m(u(\cdot,t),\partial^\bullet\underline{\varphi}_h^{n+1}(\cdot,t))\,dt=I+II+III.$$

It follows from (3.17) that

$$|I|\le\frac{c}{\tau}\int_{t^n}^{t^{n+1}}\big(\|\partial_h^\bullet\rho(\cdot,t)\|_{L^2(\Gamma)}+\|\rho(\cdot,t)\|_{L^2(\Gamma)}\big)\|\underline{\varphi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma)}dt$$

and from applying the bounds (A.4), (A.8), (A.7) and (3.23) that

$$|I|\le c\frac{h^4}{\tau}\int_{t^n}^{t^{n+1}}\|u\|^2_{H^2(\Gamma)}+\|\partial^\bullet u\|^2_{H^2(\Gamma)}\,dt+c|\varphi_h^{n+1}|^2_{n+1}.$$

Using (3.3) we find that

$$|II|=\Big|\frac{1}{\tau}\int_{t^n}^{t^{n+1}}\int_t^{t^{n+1}} a(\partial_h^\bullet u(\cdot,s),\underline{\varphi}_h^{n+1}(\cdot,s))+b(v_h(\cdot,s);u(\cdot,s),\underline{\varphi}_h^{n+1}(\cdot,s))\,dsdt\Big|$$
$$\le c\int_{t^n}^{t^{n+1}}\big(\|u(\cdot,t)\|_{H^1(\Gamma)}+\|\partial_h^\bullet u(\cdot,t)\|_{H^1(\Gamma)}\big)\,\|\nabla_\Gamma\underline{\varphi}_h^{n+1}(\cdot,t)\|_{L^2(\Gamma)}\,dt$$
$$\le c(\epsilon)\tau\int_{t^n}^{t^{n+1}}\|u\|^2_{H^2(\Gamma)}+\|\partial^\bullet u\|^2_{H^1(\Gamma)}\,dt+\epsilon|\nabla_\Gamma\varphi_h^{n+1}|^2_{n+1}.$$

For term III we use the fact that $\partial_h^\bullet\underline{\varphi}_h^{n+1}=0$ and get with (2.16)

$$|III|\le\varepsilon|\nabla_\Gamma\varphi_h^{n+1}|^2_{n+1}+c(\varepsilon)\frac{h^4}{\tau}\int_{t^n}^{t^{n+1}}\|u(\cdot,t)\|^2_{L^2(\Gamma)}\,dt.$$

The estimates for I, II, and III then imply the lemma. ∎

4.3. Proof of Theorem 2.4. We insert $\varphi_h=\theta$ in (4.4) and get

$$\partial_\tau m(\theta^n,\theta^n)-m(\theta^n,\partial_h^\bullet\theta^n)+a(\theta^{n+1},\theta^{n+1})=F_2(\theta^{n+1})-F_1(\theta^{n+1}).$$

We use Lemma 3.5 together with (3.20) and get

$$\frac{1}{2\tau}m(\theta^{n+1},\theta^{n+1})-\frac{1}{2\tau}m(\theta^n,\theta^n)-c\,m(\theta^{n+1},\theta^{n+1})+a(\theta^{n+1},\theta^{n+1})$$
$$\le|F_2(\theta^{n+1})|+|F_1(\theta^{n+1})|,$$

and the coercivity of a then leads to

$$\frac{1}{2\tau}\big(|\theta^{n+1}|^2_{n+1}-|\theta^n|^2_n\big)+c_0|\nabla_\Gamma\theta^{n+1}|^2_{n+1}\le c|\theta^{n+1}|^2_{n+1}+|F_2(\theta^{n+1})|+|F_1(\theta^{n+1})|. \tag{4.5}$$

Using the estimates for F_1 and F_2 from Lemmas 4.3 and 4.4 and summing we find, after a suitable choice of $\varepsilon > 0$,

$$\begin{aligned}|\theta^n|_n^2 + c_0\tau\sum_{k=1}^n |\nabla_\Gamma\theta^k|_k^2 &\le |\theta^0|_0^2 + c\tau\sum_{k=1}^n|\theta^k|_k^2\\ &+ch^4\int_0^{t^n} \|u\|^2_{H^2(\Gamma)} + \|\partial^\bullet u\|^2_{H^2(\Gamma)} + \|\partial_h^\bullet u_h^L\|^2_{L^2(\Gamma)} + \|u_h^L\|^2_{H^1(\Gamma)}\,dt\\ &+ch^4\tau\sum_{k=1}^n|\nabla_\Gamma u_h^k|_k^2 + c\tau^2\int_0^{t^n}\|u\|^2_{H^2(\Gamma)} + \|\partial^\bullet u\|^2_{H^1(\Gamma)}\,dt.\end{aligned}$$

The theorem now follows by a Gronwall argument, the error decomposition together with the bounds for the Ritz projection, and assumption (2.24) on the initial condition together with our stability estimates from Lemma 4.1.

Appendix. Approximation results. The results in this section are taken from [6].

A.1. Geometric perturbation errors. We begin with the bounding of the geometric perturbation errors in the bilinear forms.

LEMMA A.1. *For any $(W_h(\cdot,t),\phi_h(\cdot,t)) \in S_h(t)\times S_h(t)$ with corresponding lifts $(w_h(\cdot,t),\varphi_h(\cdot,t)) \in S_h^l(t)\times S_h^l(t)$ the following bounds hold:*

$$|m(w_h,\varphi_h) - m_h(W_h,\phi_h)| \le ch^2\|w_h\|_{L^2(\Gamma(t))}\|\varphi_h\|_{L^2(\Gamma(t))}, \tag{A.1}$$

$$|a(w_h,\varphi_h) - a_h(W_h,\phi_h)| \le ch^2\|\nabla_\Gamma w_h\|_{L^2(\Gamma(t))}\|\nabla_\Gamma\varphi_h\|_{L^2(\Gamma(t))}, \tag{A.2}$$

$$|g(v;w_h,\varphi_h) - g_h(V_h;W_h,\phi_h)| \le ch^2\|w_h\|_{H^1(\Gamma(t))}\|\varphi_h\|_{H^1(\Gamma(t))}. \tag{A.3}$$

Note that (A.3) holds with v replaced by v_h.

A.2. Approximation errors. The following result gives estimates for the approximation of the continuous material derivative by the discrete material derivative.

LEMMA A.2. *The following bounds hold for the material derivatives on* $\Gamma(t)$:

$$\|\partial^\bullet z - \partial_h^\bullet z\|_{L^2(\Gamma)} \le ch^2\|z\|_{H^1(\Gamma)}, \qquad z\in H^1(\Gamma), \tag{A.4}$$

$$\|\nabla_\Gamma(\partial^\bullet z - \partial_h^\bullet z)\|_{L^2(\Gamma)} \le ch\|z\|_{H^2(\Gamma)}, \qquad z\in H^2(\Gamma). \tag{A.5}$$

It is convenient in the error analysis to use the Ritz projection $\mathcal{R}_h : H^1(\Gamma)\to S_h^l$ defined as follows: For $z\in H^1(\Gamma)$, $\int_\Gamma z = 0$,

$$a(\mathcal{R}_h z,\varphi_h) = a(z,\varphi_h) \qquad \forall\varphi_h\in S_h^l \tag{A.6}$$

and $\int_{\Gamma_h}\mathcal{R}_h z = 0$.

LEMMA A.3. *The error in the Ritz projection satisfies the bounds*

$$\|z-\mathcal{R}_h z\|_{L^2(\Gamma)} + h\|\nabla_\Gamma(z-\mathcal{R}_h z)\|_{L^2(\Gamma)} \le ch^2\|z\|_{H^2(\Gamma)}, \tag{A.7}$$

$$\begin{aligned}&\|\partial^\bullet z - \partial^\bullet\mathcal{R}_h z\|_{L^2(\Gamma)} + h\|\nabla_\Gamma(\partial^\bullet z - \partial^\bullet\mathcal{R}_h z)\|_{L^2(\Gamma)}\\ &\qquad\le ch^2\left(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{H^2(\Gamma)}\right)\end{aligned} \tag{A.8}$$

if $h\le h_0$.

REFERENCES

[1] D. Adalsteinsson and J. A. Sethian, *Transport and diffusion of material quantities on propagating interfaces via level set methods*, J. Comput. Phys., 185 (2003), pp. 271–288.

[2] R. Barreira, C. Elliott, and A. Madzvamuse, *The surface finite element method for pattern formation on evolving biological surfaces*, J. Math. Biol., 63, (2011), pp. 1095–1119.

[3] J. W. Cahn, P. Fife, and O. Penrose, *A phase-field model for diffusion-induced grain-boundary motion*, Acta Materialia, 45 (1997), pp. 4397–4413.

[4] G. Dziuk and C. M. Elliott, *Finite elements on evolving surfaces*, IMA J. Numer. Anal., 25 (2007), pp. 385–407.

[5] G. Dziuk and C. M. Elliott, *An Eulerian approach to transport and diffusion on evolving implicit surfaces*, Comput. Vis. Sci., 13 (2010), pp. 17–28.

[6] G. Dziuk and C. M. Elliott, L^2 *estimates for the evolving surface finite element method*, Math. Comp., to appear.

[7] C. Eilks and C. M. Elliott, *Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method*, J. Comput. Phys., 227 (2008), pp. 9727–9741.

[8] C. M. Elliott and B. Stinner, *Modeling and computation of two phase geometric biomembranes using surface finite elements*, J. Comput. Phys., 229 (2010), pp. 6585–6612.

[9] C. M. Elliott, B. Stinner, V. Styles, and R. Welford, *Numerical computation of advection and diffusion on evolving diffuse interfaces*, IMA J. Numer. Anal., 31 (2011), pp. 786–812.

[10] S. Ganesan and L. Tobiska, *A coupled arbitrary Lagrangian Eulerian and Lagrangian method for computation of free-surface flows with insoluble surfactants*, J. Comput. Phys., 228 (2009), pp. 2859–2873.

[11] G. Dziuk, C. Lubich, and D. Mansour, *Runge-Kutta time discretization of parabolic differential equations on evolving surfaces*, IMA J. Numer. Anal., 32 (2011), pp. 394–416.

[12] A. J. James and J. Lowengrub, *A surfactant-conserving volume-of-fluid method for interfacial flows with insoluble surfactant*, J. Comput. Phys., 201 (2004), pp. 685–722.

[13] M. Lenz, S. F. Nemadjieu, and M. Rumpf, *A convergent finite volume scheme for diffusion on evolving surfaces*, SIAM J. Numer. Anal., 49 (2011), pp. 15–37.

[14] M. P. Neilson, J. A. Mackenzie, S. D. Webb, and R. H. Insall, *Modelling cell movement and chemotaxis using pseudopod-based feedback*, SIAM J. Sci. Comput., 33 (2011), pp. 1035–1057.

[15] K. E. Teigen, X. Li, J. Lowengrub, F. Wang, and A. Voigt, *A diffuse-interface approach for modeling transport, diffusion and adsorption/desorption of material quantities on a deformable interface*, Commun. Math. Sci., 7 (2009), pp. 1009–1037.

[16] X. Wang and Q. Du, *Modelling and simulations of multi-component lipid membranes and open membranes via diffuse interface approaches*, J. Math. Biol., 56 (2008), pp. 347–371.

[17] J.-J. Xu and H.-K. Zhao, *An Eulerian formulation for solving partial differential equations along a moving interface*, J. Sci. Comput., 19 (2003), pp. 573–594.