

An abstract framework for parabolic PDEs on evolving spaces

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Abstract

We present an abstract framework for treating the theory of well-posedness of solutions to abstract parabolic partial differential equations on evolving Hilbert spaces. This theory is applicable to variational formulations of PDEs on evolving spatial domains including moving hypersurfaces. We formulate an appropriate time derivative on evolving spaces called the material derivative and define a weak material derivative in analogy with the usual time derivative in fixed domain problems; our setting is abstract and not restricted to evolving domains or surfaces. Then we show well-posedness to a certain class of parabolic PDEs under some assumptions on the parabolic operator and the data. We finish by applying this theory to a surface heat equation, a bulk equation, a novel coupled bulk-surface system and a dynamic boundary condition problem for a moving domain.

2010 Mathematics Subject Classification: Primary 35K05, 35K90; Secondary 46E35.

Keywords: parabolic equations on moving domains, advection-diffusion on evolving surfaces

Acknowledgements

Two of the authors (A.A. and C.M.E.) were participants of the Isaac Newton Institute programme *Free Boundary Problems and Related Topics* (January – July 2014) when this article was completed. A.A. was supported by the Engineering and Physical Sciences Research Council (EPSRC) Grant EP/H023364/1 within the MASDOC Centre for Doctoral Training.

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1 Introduction

Partial differential equations on evolving or moving domains are an emerging area of research [8, 13, 27, 28], partly because their study leads to some interesting analysis but also because models describing applications such as biological phenomena can be better formulated on evolving domains (including hypersurfaces) rather than on stationary domains.

One aspect of such equations to consider is how to formulate the space of functions that have domains which evolve in time. Taking a disjoint union of the domains in time to form a non-cylindrical set is customary: see [4, 36, 28] for example. Of particular interest is [22] where the problem of a semilinear heat equation on a time-varying domain is considered; the set-up of the evolution of the domains is comparable to ours and similar function space results are shown (in the setting of Sobolev spaces). In [3], the authors define Bochner-type spaces by considering a continuous distribution of domains $\{\Gamma(t)\}_{t \in [0, T]} \subset \mathbb{R}^n$ that are embedded in a larger domain Γ . The aim of our work is to accommodate not only evolving domains but arbitrary evolving spaces. Our method, which follows that of [34], is somewhat different to the aforementioned and contains an attachment to standard Bochner spaces in a fundamental way.

A common procedure for showing well-posedness of equations on evolving domains involves a transformation of the PDE onto a fixed reference domain to which abstract techniques from functional analysis are applied [25, 29, 1, 34]. For example, in [34] the heat equation

$$\dot{u}(t) - \Delta_{\Gamma(t)}u(t) + u(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) = f(t) \quad \text{in } H^{-1}(\Gamma(t)) \quad (1.1)$$

on an evolving surface $\{\Gamma(t)\}_{t \in [0, T]}$ is considered, with \mathbf{w} representing the velocity field. The equation is pulled back onto a reference domain $\Gamma(s)$ and standard results on linear parabolic PDEs are applied. A Faedo–Galerkin method (see [2] for a historical overview of the Faedo–Galerkin method) is used in [29] (for a different PDE), where the evolving domain is represented by the evolution of a perturbation of the reference domain and *a priori* estimates are derived for a linearised problem. An adapted Galerkin method that uses the pushforward of eigenfunctions of the Laplace–Beltrami operator on $\Gamma(0)$ to form a countable dense subset of $H^1(\Gamma(t))$ is employed in [11] for the advection-diffusion equation (1.1). We abstract this approach for one of our results. Well-posedness for the same class of equations is obtained in [27] by employing a variational formulation on space-time surfaces and utilising a standard generalisation of the classical Lax–Milgram theorem used by Lions for parabolic equations. We also employ this Lions–Lax–Milgram approach in our abstract setting.

As we have seen, there is literature in which certain equations on evolving domains are studied, however, to the best of our knowledge, there is no unifying theory or framework that treats parabolic PDEs on *abstract* evolving spaces. The main aim of this paper is to provide this abstract framework. More specifically, given a linear time-dependent operator $\mathcal{A}(t)$ we study well-posedness of parabolic

problems of the form

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \quad (1.2)$$

as an equality in $V^*(t)$, with $V(t)$ a Hilbert space for each $t \in [0, T]$. A main feature of our work is the definition of an appropriate time derivative on evolving spaces *in an abstract setting*. When the said spaces are simply L^p spaces on curved or flat surfaces in \mathbb{R}^n that evolve in time, it is commonplace to take the material derivative

$$\dot{u}(t) = u_t(t) + \nabla u(t) \cdot \mathbf{w}(t)$$

from continuum mechanics as the natural time derivative. But when we have arbitrary spaces that may have no relationship whatsoever with \mathbb{R}^n it is not at all clear what the $\dot{u}(t)$ in (1.2) should mean. We will deal with this issue and define a material derivative and a weak material derivative for the abstract case. Our framework relies on the existence of a family of (pushforward) maps ϕ_t for $t \in [0, T]$ that allow us to map the initial spaces $V(0)$ and $H(0)$ to the spaces $V(t)$ and $H(t)$. A particular realisation of these maps ϕ_t in the case of, for example, the heat equation (1.1) takes into account the evolution of the surfaces $\Gamma(t)$ and hence ϕ_t will be a flow map defined by the velocity field \mathbf{w} .

We anticipate that our framework will benefit those working in numerical analysis since curved, flat, and evolving surfaces can all be treated with the same abstract procedure.

1.1 Outline

In §2, we start by setting up the function spaces and definitions required for the analysis and indeed the *statement* of equations of the form (1.2). We state our assumptions on the evolution of the domain and define strong and weak material derivatives (in analogy with the usual derivative and weak derivative utilised in fixed domain problems).

In §3 we precisely formulate the problem (1.2) that we consider and list the assumptions we make on \mathcal{A} . Statements of the main theorems of existence and uniqueness of solutions are given. The proof of one of these theorems is presented in §4. There, we make use of the generalised Lax–Milgram theorem. In §5 we formulate an adapted abstract Galerkin method similar to one described in [11] and use it to prove a regularity result. Finally, §6 contains applications of the abstract theory; firstly to a surface advection-diffusion equation, secondly to a bulk equation, then to a coupled bulk-surface system and finally to a dynamic boundary problem involving an elliptic equation on an evolving domain in \mathbb{R}^n on the boundary of which resides a parabolic PDE.

1.2 Notation and conventions

Here and below we fix $T \in (0, \infty)$. When we write expressions such as $\phi_{(\cdot)}u(\cdot)$, our intention usually (but not always) is that both of the dots (\cdot) denote the same argument; for example, $\phi_{(\cdot)}u(\cdot)$

will come to mean the map

$$t \mapsto \phi_t u(t).$$

We may reuse the same constants in calculations multiple times if their exact value is not relevant. Integrals will usually be written as $\int_0^T f(t)$ instead of $\int_0^T f(t) dt$; the latter we use only when there is ambiguity about which terms are being integrated and which are not. Finally, we shall make use of standard notation for Bochner spaces; for example, see [18, §5.9].

2 Function spaces

As we mentioned above, in order to properly understand and express the equation (1.2), we need to devise appropriate spaces of functions. First, we begin with recalling some standard results regarding Sobolev spaces from parabolic theory for the reader's convenience.

2.1 Standard Sobolev space theory

Let \mathcal{V} and \mathcal{H} be Hilbert spaces and let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ be a Hilbert triple (i.e., all embeddings are continuous and dense and \mathcal{H} is identified with its dual via the Riesz representation theorem). Recall that $u \in L^2(0, T; \mathcal{V})$ is said to have a *weak derivative* $u' \in L^2(0, T; \mathcal{V}^*)$ if there exists $w \in L^2(0, T; \mathcal{V}^*)$ such that

$$\int_0^T \zeta'(t)(u(t), v)_H = - \int_0^T \zeta(t) \langle w, v \rangle_{\mathcal{V}^*, \mathcal{V}} \quad \text{for all } \zeta \in \mathcal{D}(0, T) \text{ and } v \in V. \quad (2.1)$$

By $\mathcal{D}(0, T)$ we refer to the space of infinitely differentiable functions with compact support in $(0, T)$. We shall also make use of $\mathcal{D}([0, T]; \mathcal{V})$; this is the space of \mathcal{V} -valued infinitely differentiable functions compactly supported in the *closed* interval $[0, T]$. A helpful characterisation of this space, from Lemma 25.1 in [35, §IV.25, p. 393] is that $\mathcal{D}([0, T]; \mathcal{V})$ is the restriction $\mathcal{D}((-\infty, \infty); \mathcal{V})|_{[0, T]}$ (the restriction to $[0, T]$ of infinitely differentiable \mathcal{V} -valued functions with compact support).

Lemma 2.1. The space

$$\mathcal{W}(\mathcal{V}, \mathcal{V}^*) = \{u \in L^2(0, T; \mathcal{V}) \mid u' \in L^2(0, T; \mathcal{V}^*)\}$$

with inner product

$$(u, v)_{\mathcal{W}(\mathcal{V}, \mathcal{V}^*)} = \int_0^T (u(t), v(t))_{\mathcal{V}} + \int_0^T (u'(t), v'(t))_{\mathcal{V}^*}$$

is a Hilbert space. Furthermore,

1. The embedding $\mathcal{W}(\mathcal{V}, \mathcal{V}^*) \subset C([0, T]; \mathcal{H})$ is continuous.
2. The embedding $\mathcal{D}([0, T]; \mathcal{V}) \subset \mathcal{W}(\mathcal{V}, \mathcal{V}^*)$ is dense.

3. For $u, v \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*)$, the map $t \mapsto (u(t), v(t))_{\mathcal{H}}$ is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(u(t), v(t))_{\mathcal{H}} = \langle u'(t), v(t) \rangle_{\mathcal{V}^*, \mathcal{V}} + \langle u(t), v'(t) \rangle_{\mathcal{V}, \mathcal{V}^*}$$

for almost every $t \in [0, T]$, hence the formula of partial integration

$$(u(T), v(T))_{\mathcal{H}} - (u(0), v(0))_{\mathcal{H}} = \int_0^T \langle u'(t), v(t) \rangle_{\mathcal{V}^*, \mathcal{V}} + \int_0^T \langle u(t), v'(t) \rangle_{\mathcal{V}, \mathcal{V}^*}$$

holds.

Proof. The density result is Theorem 2.1 in [24, §1.2, p. 14]. For the rest, consult Proposition 1.2 and Corollary 1.1 in [33, §III.1, p. 106]. \square

We can characterise the weak derivative in terms of vector-valued test functions. This is useful because it more closely resembles the weak material derivative that we shall define later on.

Theorem 2.2 (Alternative characterisation of the weak derivative). The weak derivative condition (2.1) is equivalent to

$$\int_0^T (u(t), \psi'(t))_{\mathcal{H}} = - \int_0^T \langle u'(t), \psi(t) \rangle_{\mathcal{V}^*, \mathcal{V}} \quad \text{for all } \psi \in \mathcal{D}((0, T); \mathcal{V}).$$

We finish this subsection with some words on measurability.

Definition 2.3 (Weak measurability). Let X be a Hilbert space. A function $f: [0, T] \rightarrow X$ is *weakly measurable* if for every $x \in X$, the map

$$t \mapsto (f(t), x)_X$$

is measurable on $[0, T]$.

Strong measurability implies weak measurability. If the Hilbert space X turns out to be separable, then both notions of measurability are equivalent thanks to Pettis' theorem (see Theorem 1.34 in [30, §1.5, p. 22]).

2.2 Evolving spaces

Now we shall define Bochner-type function spaces to treat evolving spaces. We start with some notation and concepts on the evolution itself. We informally identify a family of Hilbert spaces $\{X(t)\}_{t \in [0, T]}$ with the symbol X , and given a family of maps $\phi_t: X_0 \rightarrow X(t)$ we say that the pair $(X, (\phi_{(\cdot)})_{t \in [0, T]})$ is **compatible** provided the following holds.

Definition 2.4 (Compatibility). For each $t \in [0, T]$, let $X(t)$ be a real separable Hilbert space with $X_0 := X(0)$, and assume that there exist linear homeomorphisms

$$\phi_t: X_0 \rightarrow X(t)$$

such that ϕ_0 is the identity. We denote by $\phi_{-t}: X(t) \rightarrow X_0$ the inverse of ϕ_t . We call ϕ_t and ϕ_{-t} the pushforward and pullback maps respectively. Furthermore, we will assume that there exists a constant C_X independent of $t \in [0, T]$ such that

$$\begin{aligned} \|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X_0} & \forall u \in X_0 \\ \|\phi_{-t} u\|_{X_0} &\leq C_X \|u\|_{X(t)} & \forall u \in X(t). \end{aligned}$$

Finally, we assume that the map

$$t \mapsto \|\phi_t u\|_{X(t)} \quad \forall u \in X_0$$

is continuous. Under these conditions, we say that $(X, (\phi_{(\cdot)})_{t \in [0, T]})$ is *compatible*. We often write the pair as $(X, \phi_{(\cdot)})$ for convenience.

In the following we will assume compatibility of $(X, \phi_{(\cdot)})$. As a consequence of these assumptions, we have that the dual operator of ϕ_t , denoted $\phi_t^*: X^*(t) \rightarrow X_0^*$, is itself a linear homeomorphism, as is its inverse $\phi_{-t}^*: X_0^* \rightarrow X^*(t)$, and they satisfy

$$\begin{aligned} \|\phi_t^* f\|_{X_0^*} &\leq C_X \|f\|_{X^*(t)} & \forall f \in X^*(t) \\ \|\phi_{-t}^* f\|_{X^*(t)} &\leq C_X \|f\|_{X_0^*} & \forall f \in X_0^*. \end{aligned}$$

By separability of X_0 , it also follows that the map

$$t \mapsto \|\phi_{-t}^* f\|_{X^*(t)} \quad \forall f \in X_0^*$$

is measurable.

Remark 2.5. Note that the above implies the equivalence of norms

$$\begin{aligned} C_X^{-1} \|u\|_{X_0} &\leq \|\phi_t u\|_{X(t)} \leq C_X \|u\|_{X_0} & \forall u \in X_0, \\ C_X^{-1} \|f\|_{X^*(t)} &\leq \|\phi_t^* f\|_{X_0^*} \leq C_X \|f\|_{X^*(t)} & \forall f \in X^*(t). \end{aligned}$$

We now define appropriate time-dependent function spaces to handle functions defined on evolving spaces. Our spaces are generalisations of those defined in [34].

Definition 2.6 (Bochner-type spaces). Define the spaces

$$\begin{aligned} L_X^2 &= \{u : [0, T] \rightarrow \bigcup_{t \in [0, T]} X(t) \times \{t\}, t \mapsto (\bar{u}(t), t) \mid \phi_{-(\cdot)} \bar{u}(\cdot) \in L^2(0, T; X_0)\} \\ L_{X^*}^2 &= \{f : [0, T] \rightarrow \bigcup_{t \in [0, T]} X^*(t) \times \{t\}, t \mapsto (\bar{f}(t), t) \mid \phi_{(\cdot)}^* \bar{f}(\cdot) \in L^2(0, T; X_0^*)\}. \end{aligned}$$

More precisely, these spaces consist of equivalence classes of functions agreeing almost everywhere in $[0, T]$, just like ordinary Bochner spaces.

We first show that these spaces are inner product spaces, and later we prove that they are in fact Hilbert spaces. For $u \in L_X^2$, we will make an abuse of notation and identify $u(t) = (\bar{u}(t), t)$ with $\bar{u}(t)$ (and likewise for $f \in L_{X^*}^2$).

Theorem 2.7. The spaces L_X^2 and $L_{X^*}^2$ are inner product spaces with the inner products

$$\begin{aligned} (u, v)_{L_X^2} &= \int_0^T (u(t), v(t))_{X(t)} dt \\ (f, g)_{L_{X^*}^2} &= \int_0^T (f(t), g(t))_{X^*(t)} dt. \end{aligned} \tag{2.2}$$

Proof. We follow the proof of [34, Lemma 3.4]. It is easy to verify that the expressions in (2.2) define inner products if the integrals on the right hand sides are well-defined, which we now check.

To show that the integrand $(u(t), v(t))_{X(t)}: [0, T] \rightarrow \mathbb{R}$ is integrable for all $u, v \in L_X^2$, by the parallelogram law, it suffices to show that $\|u(t)\|_{X(t)}^2$ is integrable for all $u \in L_X^2$. Since $u \in L_X^2$, $\phi_{-\cdot}u(\cdot) \in L^2(0, T; X_0)$, and therefore there exists a sequence of measurable simple functions

$$\tilde{u}_n(t) = \sum_{i=1}^{M_n} u_{i,n} \mathbf{1}_{B_i}(t)$$

where $u_{i,n} \in X_0$ and the $B_i \subset [0, T]$ are measurable, disjoint and partition $[0, T]$, such that \tilde{u}_n converges to $\phi_{-\cdot}u(\cdot)$ in $L^2(0, T; X_0)$. From this it follows that $\|\tilde{u}_n(t) - \phi_{-t}u(t)\|_{X_0}^2 \rightarrow 0$ a.e. for a subsequence which we labelled as \tilde{u}_n again. By the continuity of ϕ_t , we have $\phi_t \tilde{u}_n(t) \rightarrow u(t)$ in $X(t)$ pointwise a.e., hence

$$\|\phi_t \tilde{u}_n(t)\|_{X(t)} \rightarrow \|u(t)\|_{X(t)}.$$

We have

$$\|\phi_t \tilde{u}_n(t)\|_{X(t)}^2 = \left\| \phi_t \left(\sum_{i=1}^{M_n} u_{i,n} \mathbf{1}_{B_i}(t) \right) \right\|_{X(t)}^2 = \sum_{i=1}^{M_n} \|\phi_t u_{i,n}\|_{X(t)}^2 \mathbf{1}_{B_i} \tag{2.3}$$

where the last equality follows by linearity and the fact that $\mathbf{1}_{B_i} \mathbf{1}_{B_j} = 0$ for $i \neq j$ (since the B_i are disjoint).

By assumption, the $\|\phi_t u_{i,n}\|_{X(t)}$ are measurable functions with respect to t , so (2.3) is measurable too. The pointwise limit of measurable functions is measurable, hence $\|u(t)\|_{X(t)}$ is measurable. Finally, since

$$\|u(t)\|_{X(t)} \leq C_X \|\phi_{-t}u(t)\|_{X_0},$$

$\|u(t)\|_{X(t)}$ is square-integrable.

This proves the theorem for L_X^2 . The process is the same for the case of $L_{X^*}^2$ except we replace ϕ_{-t} and ϕ_t with the dual maps ϕ_t^* and ϕ_{-t}^* . \square

Corollary 2.8. Let $u \in L^2_X$ and $f \in L^2_{X^*}$. Then there exist simple measurable functions $u_n \in L^2(0, T; X_0)$ and $f_n \in L^2(0, T; X_0^*)$ such that for almost every $t \in [0, T]$,

$$\begin{aligned} \phi_t u_n(t) &\rightarrow u(t) && \text{in } X(t) \\ \phi_{-t}^* f_n(t) &\rightarrow f(t) && \text{in } X^*(t) \end{aligned}$$

as $n \rightarrow \infty$.

The following result is required to show that the above spaces are complete.

Lemma 2.9 (Isomorphism with standard Bochner spaces). The maps

$$\begin{aligned} u &\mapsto \phi_{(\cdot)} u(\cdot) && \text{from } L^2(0, T; X_0) \text{ to } L^2_X \\ f &\mapsto \phi_{-(\cdot)}^* f(\cdot) && \text{from } L^2(0, T; X_0^*) \text{ to } L^2_{X^*} \end{aligned}$$

are both isomorphisms between the respective spaces.

For the proof of the L^2_X case, one makes an argument similar to that in the proof of Theorem 2.7 and shows that given an arbitrary $u \in L^2(0, T; X_0)$, the map $t \mapsto \|\phi_t u(t)\|_{X(t)}^2$ is indeed measurable (and then it follows that $\|\phi_{(\cdot)} u(\cdot)\|_{L^2_X}$ is finite). That the spaces are isomorphic follows from the above (which shows that there is a map from $L^2(0, T; X_0)$ to L^2_X) and the definition of L^2_X . The isomorphism is $T: L^2(0, T; X_0) \rightarrow L^2_X$ where

$$Tu = \phi_{(\cdot)} u(\cdot) \quad \text{and} \quad T^{-1}v = \phi_{-(\cdot)} v(\cdot).$$

It is easy to check that T is linear and bijective. The proof for the $L^2_{X^*}$ case uses the same readjustments as before.

The next lemma, which is a consequence of the uniform bounds on ϕ_t and ϕ_t^* , will be in constant use throughout this work.

Lemma 2.10. The equivalence of norms

$$\begin{aligned} \frac{1}{C_X} \|u\|_{L^2_X} &\leq \|\phi_{-(\cdot)} u(\cdot)\|_{L^2(0, T; X_0)} \leq C_X \|u\|_{L^2_X} && \forall u \in L^2_X \\ \frac{1}{C_X} \|f\|_{L^2_{X^*}} &\leq \|\phi_{(\cdot)}^* f(\cdot)\|_{L^2(0, T; X_0^*)} \leq C_X \|f\|_{L^2_{X^*}} && \forall f \in L^2_{X^*} \end{aligned}$$

holds.

Corollary 2.11. The spaces L^2_X and $L^2_{X^*}$ are Hilbert spaces.

Proof. Since L^2_X and $L^2(0, T; X_0)$ are isomorphic and the latter space is complete, so too is L^2_X by the equivalence of norms result in the previous lemma. \square

We now investigate the relationship between the dual space of L^2_X and the space $L^2_{X^*}$. We in fact prove that these spaces can be identified; this requires the following preliminary lemmas.

Lemma 2.12. For $f \in L^2_{X^*}$ and $u \in L^2_X$, the map

$$t \mapsto \langle f(t), u(t) \rangle_{X^*(t), X(t)}$$

is integrable on $[0, T]$.

Proof. According to Corollary 2.8, there are simple measurable functions $f_n \in L^2(0, T; X_0^*)$ and $u_n \in L^2(0, T; X_0)$ that satisfy

$$\begin{aligned} \phi_{-t}^* f_n(t) &\rightarrow f(t) && \text{in } X^*(t) \text{ for a.e. } t \in [0, T] \\ \phi_t u_n(t) &\rightarrow u(t) && \text{in } X(t) \text{ for a.e. } t \in [0, T] \end{aligned}$$

as $n \rightarrow \infty$. We have the convergence

$$|\langle f(t), u(t) \rangle_{X^*(t), X(t)} - \langle \phi_{-t}^* f_n(t), \phi_t u_n(t) \rangle_{X^*(t), X(t)}| \rightarrow 0$$

(by adding and subtracting $\langle \phi_{-t}^* f_n(t), u(t) \rangle_{X^*(t), X(t)}$ on the left hand side) for almost every $t \in [0, T]$. So, $\langle f(t), u(t) \rangle_{X^*(t), X(t)}$ being the pointwise a.e. limit of measurable functions is itself measurable. That its integral is finite is trivial. \square

Lemma 2.13. Suppose that $f(t) \in X^*(t)$ for almost every $t \in [0, T]$ with

$$\int_0^T \|f(t)\|_{X^*(t)}^2 < \infty,$$

and that for every $u \in L^2_X$, the map

$$t \mapsto \langle f(t), u(t) \rangle_{X^*(t), X(t)}$$

is measurable. Then $f \in L^2_{X^*}$.

Proof. We rewrite

$$\langle f(t), u(t) \rangle_{X^*(t), X(t)} = \langle \phi_{-t}^* \phi_t^* f(t), u(t) \rangle_{X^*(t), X(t)} = \langle \phi_t^* f(t), \phi_{-t} u(t) \rangle_{X_0^*, X_0}.$$

Now, the left hand side is measurable, hence the map

$$t \mapsto \langle \phi_t^* f(t), \phi_{-t} u(t) \rangle_{X_0^*, X_0}$$

is measurable on $[0, T]$ for every $u \in L^2_X$.

Given $w \in X_0$, the element $u(\cdot) = \phi_{(\cdot)} w \in L^2_X$, so we have (from Definition (2.3) or Footnote 80 in [31, §1.4, p. 36] for example) that $\phi_{(\cdot)}^* f(\cdot): [0, T] \rightarrow X_0^*$ is weakly measurable.

Now, as remarked after Definition 2.3, we use Pettis' theorem to conclude that $\phi_{(\cdot)}^* f(\cdot)$ is indeed strongly measurable. Hence we can compute

$$\|\phi_{(\cdot)}^* f(\cdot)\|_{L^2(0, T; X_0^*)}^2 = \int_0^T \|\phi_t^* f(t)\|_{X_0^*}^2 \leq C_X^2 \int_0^T \|f(t)\|_{X^*(t)}^2 < \infty,$$

so $\phi_{(\cdot)}^* f(\cdot) \in L^2(0, T; X_0^*)$, giving $f \in L^2_{X^*}$. \square

Lemma 2.14 (Identification of $(L_X^2)^*$ and $L_{X^*}^2$). The spaces $(L_X^2)^*$ and $L_{X^*}^2$ are isometrically isomorphic. Hence, we identify $(L_X^2)^* \equiv L_{X^*}^2$, and the dual pairing of $f \in L_{X^*}^2$ with $u \in L_X^2$ is

$$\langle f, u \rangle_{L_{X^*}^2, L_X^2} = \int_0^T \langle f(t), u(t) \rangle_{X^*(t), X(t)} dt.$$

Proof. Define the linear map $\mathcal{J}: L_{X^*}^2 \rightarrow (L_X^2)^*$ by

$$\langle \mathcal{J}f, \cdot \rangle_{(L_X^2)^*, L_X^2} = \int_0^T \langle f(t), (\cdot)(t) \rangle_{X^*(t), X(t)} dt.$$

This is well-defined due to Lemma 2.12. We must check that \mathcal{J} is an isometric isomorphism.

Suppose that $F \in (L_X^2)^*$. We first need to show that there exists a unique $f \in L_{X^*}^2$ such that $\mathcal{J}f = F$. To do this, we use the Riesz map $\mathcal{R}: (L_X^2)^* \rightarrow L_X^2$ to write

$$\langle F, u \rangle_{(L_X^2)^*, L_X^2} = (\mathcal{R}F, u)_{L_X^2} = \int_0^T (\mathcal{R}F(t), u(t))_{X(t)}, \quad (2.4)$$

and then with $\mathcal{S}_t^{-1}: X(t) \rightarrow X^*(t)$ denoting the Riesz map on $X(t)$, we get

$$(\mathcal{R}F(t), u(t))_{X(t)} = \langle \mathcal{S}_t^{-1}(\mathcal{R}F(t)), u(t) \rangle_{X^*(t), X(t)}.$$

Now, from (2.4), the right hand side of this equality must be integrable. Hence

$$t \mapsto \langle \mathcal{S}_t^{-1}(\mathcal{R}F(t)), u(t) \rangle_{X^*(t), X(t)}$$

is measurable for every $u \in L_X^2$. Now, the question is whether $\mathcal{S}_{(\cdot)}^{-1}(\mathcal{R}F(\cdot)) \in L_{X^*}^2$. We want to use Lemma 2.13 so we need to check its hypotheses. Clearly $\mathcal{S}_t^{-1}(\mathcal{R}F(t)) \in X^*(t)$, and by the isometry of the Riesz maps,

$$\int_0^T \|\mathcal{S}_t^{-1}(\mathcal{R}F(t))\|_{X^*(t)}^2 = \int_0^T \|\mathcal{R}F(t)\|_{X(t)}^2 = \|\mathcal{R}F\|_{L_X^2}^2 = \|F\|_{(L_X^2)^*}^2 \quad (2.5)$$

which is finite. Therefore, we obtain $\mathcal{S}_{(\cdot)}^{-1}(\mathcal{R}F(\cdot)) \in L_{X^*}^2$ by Lemma 2.13. So $\mathcal{J}(\mathcal{S}_{(\cdot)}^{-1}\mathcal{R}F(\cdot)) = F$.

For uniqueness, suppose that $\mathcal{J}f_1 = \mathcal{J}f_2$. Then

$$\begin{aligned} \langle \mathcal{J}f_1 - \mathcal{J}f_2, u \rangle_{(L_X^2)^*, L_X^2} &= \int_0^T \langle f_1(t) - f_2(t), u(t) \rangle_{X^*(t), X(t)} \\ &= \int_0^T \langle \phi_t^*(f_1(t) - f_2(t)), \phi_{-t}u(t) \rangle_{X_0^*, X_0} \\ &= \langle \phi_{(\cdot)}^*(f_1(\cdot) - f_2(\cdot)), \hat{u} \rangle_{L^2(0, T; X_0^*), L^2(0, T; X_0)}, \quad (\text{with } \hat{u} = \phi_{-(\cdot)}u(\cdot)) \end{aligned}$$

which holds for all $\hat{u} \in L^2(0, T; X_0)$. This implies that $f_1 = f_2$.

To see that \mathcal{J} is an isometry, we define $\mathcal{J}^{-1}: (L_X^2)^* \rightarrow L_{X^*}^2$ by $\mathcal{J}^{-1}F = \mathcal{S}_{(\cdot)}^{-1}\mathcal{R}F(\cdot)$ and use (2.5) to conclude. \square

The next lemma is easy to prove using Lemma 2.10.

Lemma 2.15. The spaces L_X^2 and $L_{X^*}^2$ are separable.

Although we have no notion of continuity in time for a function $u \in L_X^2$, we can nevertheless make the following definition.

Definition 2.16 (Spaces of pushed-forward continuously differentiable functions). Define

$$\begin{aligned} C_X^k &= \{\xi \in L_X^2 \mid \phi_{-(\cdot)}\xi(\cdot) \in C^k([0, T]; X_0)\} \quad \text{for } k \in \{0, 1, \dots\} \\ \mathcal{D}_X(0, T) &= \{\eta \in L_X^2 \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}((0, T); X_0)\} \\ \mathcal{D}_X[0, T] &= \{\eta \in L_X^2 \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}([0, T]; X_0)\}. \end{aligned}$$

Since $\mathcal{D}((0, T); X_0) \subset \mathcal{D}([0, T]; X_0)$, we have

$$\mathcal{D}_X(0, T) \subset \mathcal{D}_X[0, T] \subset C_X^k.$$

2.3 Evolving Hilbert space triple structure

In the preceding, we set up the Hilbert space L_X^2 and its dual $L_{X^*}^2$ based on an arbitrary family of separable Hilbert spaces $\{X(t)\}_{t \in [0, T]}$ and a suitable family of maps $\{\phi_t\}_{t \in [0, T]}$. In standard PDE theory, often there is a notion of a Hilbert triple involved in the formulation of the problem. We now lay the analogous groundwork for posing PDEs on evolving spaces. For each $t \in [0, T]$, let $V(t)$ and $H(t)$ be (real) separable Hilbert spaces with $V_0 := V(0)$ and $H_0 := H(0)$. Let $V(t) \subset H(t)$ be continuously and densely embedded. Identifying $H(t)$ with its dual space $H^*(t)$ via the Riesz representation theorem, it then follows that $H(t) \subset V^*(t)$ is also continuous and dense. In other words,

$$V(t) \subset H(t) \subset V^*(t)$$

is a Hilbert triple. We often make use of the identification

$$\langle f, u \rangle_{V^*(t), V(t)} = (f, u)_{H(t)} \quad \text{whenever } f \in H(t) \text{ and } u \in V(t).$$

Assumptions 2.17. We will assume compatibility in the sense of Definition 2.4 for the family $\{H(t)\}_{t \in [0, T]}$ and a family of linear homeomorphisms $\{\phi_t\}_{t \in [0, T]}$; that is, we assume $(H, \phi_{(\cdot)})$ is a compatible pair. In addition, we also assume that $(V, \phi_{(\cdot)}|_{V_0})$ is compatible. We will simply write ϕ_t instead of $\phi_t|_{V_0}$, and we will denote the dual operator of $\phi_t: V_0 \rightarrow V(t)$ by $\phi_t^*: V^*(t) \rightarrow V_0^*$; we are not interested in the dual of $\phi_t: H_0 \rightarrow H(t)$.

Let us summarise the meaning of these assumptions below for the convenience of the reader.

1. For each $t \in [0, T]$, there exists a linear homeomorphism

$$\phi_t: H_0 \rightarrow H(t)$$

such that ϕ_0 is the identity.

2. The restriction $\phi_t|_{V_0}$ (which we will denote by ϕ_t) is also a linear homeomorphism from V_0 to $V(t)$.

3. There exist constants C_H and C_V independent of $t \in [0, T]$ such that

$$\begin{aligned} \|\phi_t u\|_{H(t)} &\leq C_H \|u\|_{H_0} & \forall u \in H_0, \\ \|\phi_t u\|_{V(t)} &\leq C_V \|u\|_{V_0} & \forall u \in V_0. \end{aligned}$$

4. We will only be interested in the dual of $\phi_t: V_0 \rightarrow V(t)$, denoted by $\phi_t^*: V^*(t) \rightarrow V_0^*$, which satisfies

$$\|\phi_t^* f\|_{V_0^*} \leq C_V \|f\|_{V^*(t)} \quad \forall f \in V^*(t).$$

5. The inverses of ϕ_t and ϕ_t^* will be denoted by ϕ_{-t} and ϕ_{-t}^* respectively, and these are uniformly bounded:

$$\begin{aligned} \|\phi_{-t} u\|_{H_0} &\leq \tilde{C}_H \|u\|_{H(t)} & \forall u \in H(t), \\ \|\phi_{-t} u\|_{V_0} &\leq \tilde{C}_V \|u\|_{V(t)} & \forall u \in V(t), \\ \|\phi_{-t}^* f\|_{V^*(t)} &\leq \tilde{C}_V \|f\|_{V_0^*} & \forall f \in V_0^*. \end{aligned}$$

6. The maps

$$\begin{aligned} t &\mapsto \|\phi_t u\|_{H(t)} & \forall u \in H_0 \\ t &\mapsto \|\phi_t u\|_{V(t)} & \forall u \in V_0 \end{aligned}$$

are continuous, and the map

$$t \mapsto \|\phi_{-t}^* f\|_{V^*(t)} \quad \forall f \in V_0^*$$

is measurable.

Our work in §2.2 tells us (amongst other things) that the spaces L^2_H , L^2_V , and $L^2_{V^*}$ are Hilbert spaces with the inner product given by the formula (2.2).

Remark 2.18. These homeomorphisms ϕ_t are similar to Arbitrary Lagrangian Eulerian (ALE) maps that are ubiquitous in applications on moving domains. See [1] for an account of the ALE framework and a comparable set-up.

By the density of $L^2(0, T; V_0)$ in $L^2(0, T; H_0)$, we obtain the next result.

Lemma 2.19. The space L^2_V is dense in L^2_H .

The previous lemma tells us that L^2_V embeds into L^2_H densely, and it is obvious that the embedding is continuous. Therefore, identifying L^2_H with its dual via the Riesz map, the relationship

$$L^2_V \subset L^2_H \subset L^2_{V^*}$$

is a Hilbert triple.

2.4 Abstract strong and weak material derivatives

Suppose $\{\Gamma(t)\}_{t \in [0, T]}$ is a family of (sufficiently smooth) hypersurfaces evolving with velocity field \mathbf{w} , and that for each $t \in [0, T]$, $u(t)$ is a sufficiently smooth function defined on $\Gamma(t)$. Then the appropriate time derivative of u takes into account the movement of the spatial points too, and this time derivative is known as the (strong) *material derivative*, which we can write informally as

$$\dot{u}(t, x) = \frac{d}{dt}u(t, x(t)) = u_t(t, x) + \nabla u(t, x) \cdot \mathbf{w}(t, x). \quad (2.6)$$

This is well-studied: see [6] or [7, §1.2, p. 6] for the flat case. Our aim is to generalise this material derivative to arbitrary functions and arbitrary evolving spaces (and not just merely evolving surfaces).

Definition 2.20 (Strong material derivative). For $\xi \in C_X^1$ define the *strong material derivative* $\dot{\xi} \in C_X^0$ by

$$\dot{\xi}(t) := \phi_t \left(\frac{d}{dt}(\phi_{-t}\xi(t)) \right). \quad (2.7)$$

This definition is generalised from [34]. So we see that the space C_X^1 is the space of functions with a strong material derivative, justifying the notation. In the evolving surface case, we show in §6.1 that this abstract formula agrees with (2.6). The following remark observes that the pushforward of elements of $X(0)$ into $X(t)$ have zero material derivative.

Remark 2.21. Observe that given $\eta \in X(0)$,

$$(\phi_t \dot{\eta}) = 0$$

and that for $\xi \in C_X^1$

$$\dot{\xi} = 0 \iff \exists \eta \in X(0) \text{ such that } \xi(t) = \phi_t \eta.$$

It may be the case that solutions to the PDE (1.2)

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t)$$

may not exist if we ask for $u \in C_V^1$, that is, they may not possess strong material derivatives. We can relax this and ask for \dot{u} to exist in a weaker sense, just like one does for the usual time derivative in parabolic problems on fixed domains.

Heuristically, what should such a weak material derivative satisfy? Taking a clue from Lemma 2.1, we expect

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \dot{u}(t), v(t) \rangle_{V^*(t), V(t)} + \langle \dot{v}(t), u(t) \rangle_{V^*(t), V(t)} + \text{extra term}$$

where we envisage an extra term because the Hilbert space associated with the inner product depends on t itself, and certainly we should require the integration by parts formula

$$\int_0^T \frac{d}{dt}(u(t), \eta(t))_{H(t)} = 0 \quad \forall \eta \in \mathcal{D}_V(0, T).$$

The identification of this extra term and a definition of the weak material derivative is what the rest of this section is devoted to.

Definition 2.22 (Relationship between the inner product on $H(t)$ and the space H_0). For all $t \in [0, T]$, define the bounded bilinear form $\hat{b}(t; \cdot, \cdot): H_0 \times H_0 \rightarrow \mathbb{R}$ by

$$\hat{b}(t; u_0, v_0) = (\phi_t u_0, \phi_t v_0)_{H(t)} \quad \forall u_0, v_0 \in H_0.$$

This gives us a way of pulling back the inner product on $H(t)$ onto a bilinear form on H_0 by the formula $(u, v)_{H(t)} = \hat{b}(t; \phi_{-t}u, \phi_{-t}v)$. It is also clear that $\hat{b}(0; \cdot, \cdot) = (\cdot, \cdot)_{H_0}$ by definition. In fact, one can see for each $t \in [0, T]$ that $\hat{b}(t; \cdot, \cdot)$ is an inner product on H_0 (and it is norm-equivalent with the norm on H_0); thanks to the Riesz representation theorem, there exists for each $t \in [0, T]$ a bounded linear operator $T_t: H_0 \rightarrow H_0$ such that

$$\hat{b}(t; u_0, v_0) = (T_t u_0, v_0)_{H_0} = (u_0, T_t v_0)_{H_0}. \quad (2.8)$$

Remark 2.23. It is not difficult to see that $T_t \equiv \phi_t^A \phi_t$, where $\phi_t^A: H(t) \rightarrow H_0$ denotes the Hilbert-adjoint of $\phi_t: H_0 \rightarrow H(t)$.

Assumptions 2.24. We shall assume the following for all $u_0, v_0 \in H_0$:

$$\theta(t, u_0) := \frac{d}{dt} \|\phi_t u_0\|_{H(t)}^2 \text{ exists classically} \quad (2.9)$$

$$u_0 \mapsto \theta(t, u_0) \text{ is continuous} \quad (2.10)$$

$$|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| \leq C \|u_0\|_{H_0} \|v_0\|_{H_0} \quad (2.11)$$

where the constant C is independent of $t \in [0, T]$.

We are now able to define $\hat{c}(t; \cdot, \cdot): H_0 \times H_0 \rightarrow \mathbb{R}$ by

$$\hat{c}(t; u_0, v_0) := \frac{d}{dt} \hat{b}(t; u_0, v_0) = \frac{1}{4} (\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)). \quad (2.12)$$

Denoting by $\hat{C}(t)$ the operator

$$\langle \hat{C}(t)u_0, v_0 \rangle := \hat{c}(t; u_0, v_0), \quad (2.13)$$

it follows by (2.11) that $\hat{C}(t): H_0 \rightarrow H_0^*$.

Definition 2.25 (Definition of the bilinear form $c(t; \cdot, \cdot)$). For $u, v \in H(t)$, define $c(t; \cdot, \cdot): H(t) \times H(t) \rightarrow \mathbb{R}$ by

$$c(t; u, v) = \hat{c}(t; \phi_{-t}u, \phi_{-t}v).$$

Lemma 2.26. The map

$$t \mapsto c(t; u(t), v(t)) \quad \forall u, v \in L_H^2$$

is measurable and $c(t; \cdot, \cdot): H(t) \times H(t) \rightarrow \mathbb{R}$ is bounded independently of t :

$$|c(t; u, v)| \leq C \|u\|_{H(t)} \|v\|_{H(t)}.$$

Proof. If $u, v \in L^2_H$, then by (2.12),

$$\begin{aligned} c(t; u(t), v(t)) &= \hat{c}(t; \phi_{-t}u(t), \phi_{-t}v(t)) \\ &= \frac{1}{4} (\theta(t, \phi_{-t}u(t) + \phi_{-t}v(t)) - \theta(t, \phi_{-t}u(t) - \phi_{-t}v(t))) \end{aligned}$$

and it follows that $t \mapsto c(t; u(t), v(t))$ is measurable because $t \mapsto \theta(t, \phi_{-t}w(t))$ is measurable for $w \in L^2_H$. This in turn can be seen by noticing that $\theta: [0, T] \times H_0 \rightarrow \mathbb{R}$ is a Carathéodory function: the map $t \mapsto \theta(t, x)$ is measurable and by assumption (2.10) the map $x \mapsto \theta(t, x)$ is continuous; thus by [16, Remark 3.4.2] the desired measurability is achieved. The bound on $c(t; \cdot, \cdot)$ is a consequence of the assumption (2.11). \square

Lemma 2.27. For $\sigma_1, \sigma_2 \in C^1([0, T]; H_0)$, the map

$$t \mapsto \hat{b}(t; \sigma_1(t), \sigma_2(t))$$

is differentiable in the classical sense and

$$\frac{d}{dt} \hat{b}(t; \sigma_1(t), \sigma_2(t)) = \hat{b}(t; \sigma_1'(t), \sigma_2(t)) + \hat{b}(t; \sigma_1(t), \sigma_2'(t)) + \hat{c}(t; \sigma_1(t), \sigma_2(t)).$$

This follows simply by using the definition of the derivative as a limit.

Definition 2.28 (Weak material derivative). For $u \in L^2_V$, if there exists a function $g \in L^2_{V^*}$ such that

$$\int_0^T \langle g(t), \eta(t) \rangle_{V^*(t), V(t)} = - \int_0^T (u(t), \dot{\eta}(t))_{H(t)} - \int_0^T c(t; u(t), \eta(t))$$

holds for all $\eta \in \mathcal{D}_V(0, T)$, then we say that g is the *weak material derivative* of u , and we write

$$\dot{u} = g \text{ or } \partial^\bullet u = g.$$

This concept of a weak material derivative is indeed well-defined: if it exists, it is unique, and every strong material derivative is also a weak material derivative. To prove these facts is a fairly standard exercise: for uniqueness, assume there exist two material derivatives for the same function and then linearity and the density of $\mathcal{D}((0, T); V_0)$ (the space of test functions) in $L^2(0, T; V_0)$ gives the result. To show that a strong material derivative is also a weak material derivative, one can use Lemma 2.27 and the relations between $\hat{b}(t; \cdot, \cdot)$, $b(t; \cdot, \cdot)$ and $\hat{c}(t; \cdot, \cdot)$, $c(t; \cdot, \cdot)$.

2.5 Solution space

We can now consider the spaces that solutions of our PDEs will lie in.

Definition 2.29 (The space $W(V, V^*)$). Define the solution space

$$W(V, V^*) = \{u \in L^2_V \mid \dot{u} \in L^2_{V^*}\}$$

and endow it with the inner product

$$(u, v)_{W(V, V^*)} = \int_0^T (u(t), v(t))_{V(t)} + \int_0^T (\dot{u}(t), \dot{v}(t))_{V^*(t)}.$$

We also shall require the subspaces

$$\begin{aligned} W_0(V, V^*) &= \{u \in W(V, V^*) \mid u(0) = 0\} \text{ and} \\ W(V, H) &= \{u \in L^2_V \mid \dot{u} \in L^2_H\}. \end{aligned}$$

In order to prove existence theorems, we need some properties of the space $W(V, V^*)$ which turns out to be deeply linked with the following standard Sobolev–Bochner space.

Definition 2.30 (The space $\mathcal{W}(V_0, V_0^*)$). Define

$$\mathcal{W}(V_0, V_0^*) = \{v \in L^2(0, T; V_0) \mid v' \in L^2(0, T; V_0^*)\}$$

to be the space $\mathcal{W}(\mathcal{V}, \mathcal{V}^*)$ introduced in §2.1 with $\mathcal{V} = V_0$ and $\mathcal{H} = H_0$.

It is convenient to introduce the following notion of *evolving space equivalence*.

Assumption and Definition 2.31. We assume that there is an *evolving space equivalence* between $W(V, V^*)$ and $\mathcal{W}(V_0, V_0^*)$. By this we mean that

$$v \in W(V, V^*) \quad \text{if and only if} \quad \phi_{-(\cdot)}v(\cdot) \in \mathcal{W}(V_0, V_0^*),$$

and the equivalence of norms

$$C_1 \|\phi_{-(\cdot)}v(\cdot)\|_{\mathcal{W}(V_0, V_0^*)} \leq \|v\|_{W(V, V^*)} \leq C_2 \|\phi_{-(\cdot)}v(\cdot)\|_{\mathcal{W}(V_0, V_0^*)}$$

holds.

We now show under certain conditions that this assumption holds.

Theorem 2.32. Suppose that

$$u \in \mathcal{W}(V_0, V_0^*) \quad \text{if and only if} \quad T_{(\cdot)}u(\cdot) \in \mathcal{W}(V_0, V_0^*) \tag{T1}$$

and that there exist operators

$$\hat{S}(t): V_0^* \rightarrow V_0^* \quad \text{and} \quad \hat{D}(t): V_0 \rightarrow V_0^*$$

such that for $u \in \mathcal{W}(V_0, V_0^*)$,

$$(T_t u(t))' = \hat{S}(t)u'(t) + \hat{C}(t)u(t) + \hat{D}(t)u(t) \tag{T2}$$

and

$$\hat{S}(\cdot)u'(\cdot) \in L^2(0, T; V_0^*) \quad \text{and} \quad \hat{D}(\cdot)u(\cdot) \in L^2(0, T; V_0^*).$$

Suppose also that $\hat{S}(t)$ and $\hat{D}(t)$ are bounded independently of $t \in [0, T]$, and that $\hat{S}(t)$ has an inverse $\hat{S}(t)^{-1}: V_0^* \rightarrow V_0^*$ which also is bounded independently of $t \in [0, T]$. Then $W(V, V^*)$ is equivalent to $\mathcal{W}(V_0, V_0^*)$ in the sense of Definition 2.31.

Proof. First, suppose $u \in \mathcal{W}(V_0, V_0^*)$. Clearly $\phi_{(\cdot)}u(\cdot) \in L^2_V$ and we need only to show that $\partial^\bullet(\phi_{(\cdot)}u(\cdot)) \in L^2_{V^*}$ exists.

Let $\eta \in \mathcal{D}_V(0, T)$. Consider

$$\begin{aligned}
\int_0^T (\phi_t u(t), \dot{\eta}(t))_{H(t)} &= \int_0^T (T_t u(t), (\phi_{-t} \eta(t))')_{H_0} \\
&\quad \text{(rewriting the integrand using } \hat{b}(t; \cdot, \cdot) \text{ and by (2.8))} \\
&= - \int_0^T \langle \hat{S}(t)u'(t) + \hat{C}(t)u(t) + \hat{D}(t)u(t), \phi_{-t} \eta(t) \rangle_{V_0^*, V_0} \\
&\quad \text{(by (T1) and (T2))} \\
&= - \int_0^T \langle \phi_{-t}^* (\hat{S}(t)u'(t) + \hat{D}(t)u(t)), \eta(t) \rangle_{V^*(t), V(t)} - \int_0^T c(t; \phi_t u(t), \eta(t)).
\end{aligned} \tag{2.14}$$

This shows that $\partial^\bullet(\phi_{(\cdot)}u(\cdot))$ exists.

Conversely, let $u \in W(V, V)$. We need to show the existence of $\phi_{-(\cdot)}u(\cdot)'$ in $L^2(0, T; V_0^*)$. We start with the weak material derivative condition:

$$\int_0^T \langle \dot{u}(t), \eta(t) \rangle_{V^*(t), V(t)} = - \int_0^T (u(t), \dot{\eta}(t))_{H(t)} - \int_0^T c(t; u(t), \eta(t))$$

for test functions $\eta \in \mathcal{D}_V(0, T)$. Pulling back leads to

$$\int_0^T \langle \phi_t^* \dot{u}(t), \phi_{-t} \eta(t) \rangle_{V_0^*, V_0} = - \int_0^T \hat{b}(t; \phi_{-t} u(t), (\phi_{-t} \eta(t))') - \int_0^T \hat{c}(t; \phi_{-t} u(t), \phi_{-t} \eta(t)).$$

Using (2.8) and (2.13) and rearranging:

$$\int_0^T (T_t \phi_{-t} u(t), (\phi_{-t} \eta(t))') = - \int_0^T \langle \phi_t^* \dot{u}(t) + \hat{C}(t) \phi_{-t} u(t), \phi_{-t} \eta(t) \rangle_{V_0^*, V_0}. \tag{2.15}$$

It follows that $T_{(\cdot)}\phi_{-(\cdot)}u(\cdot)$ has a weak derivative, and hence by (T1) as does $\phi_{-(\cdot)}u(\cdot)$. This proves the bijection between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$.

For the equivalence of norms, let $u \in W(V, V^*)$. From (2.14), we see that

$$\dot{u}(t) = \phi_{-t}^* (\hat{S}(t)(\phi_{-t} u(t))' + \hat{D}(t) \phi_{-t} u(t))$$

which we can bound thanks to the boundedness of $\hat{S}(t)$ and $\hat{D}(t)$:

$$\|\dot{u}(t)\|_{V(t)} \leq C \left(\|(\phi_{-t} u(t))'\|_{V_0^*} + \|\phi_{-t} u(t)\|_{V_0} \right).$$

So we have achieved

$$\|u\|_{W(V, V^*)} \leq C_2 \|\phi_{-(\cdot)}u(\cdot)\|_{\mathcal{W}(V_0, V_0^*)}.$$

For the reverse inequality, we start with the weak derivative condition

$$\begin{aligned} \int_0^T (T_t \phi_{-t} u(t), \psi'(t))_{H_0} &= - \int_0^T \langle \hat{S}(t)(\phi_{-t} u(t))' + \hat{D}(t) \phi_{-t} u(t), \psi(t) \rangle_{V_0^*, V_0} \\ &\quad - \int_0^T \langle \hat{C}(t) \phi_{-t} u(t), \psi(t) \rangle_{V_0^*, V_0}, \end{aligned}$$

where $\psi \in \mathcal{D}((0, T); V_0)$. Using the formula (2.15), we find

$$\int_0^T \langle \hat{S}(t)(\phi_{-t} u(t))', \psi(t) \rangle_{V_0^*, V_0} = \int_0^T \langle \phi_t^* \dot{u}(t) - \hat{D}(t) \phi_{-t} u(t), \psi(t) \rangle_{V_0^*, V_0}$$

which implies

$$(\phi_{-t} u(t))' = \hat{S}(t)^{-1} (\phi_t^* \dot{u}(t) - \hat{D}(t) \phi_{-t} u(t)).$$

From this we obtain a bound of the form

$$\|(\phi_{-t} u(t))'\|_{V_0^*} \leq C \left(\|\dot{u}(t)\|_{V^*(t)} + \|u(t)\|_{V(t)} \right)$$

which implies the result. \square

Corollary 2.33. The space $W(V, V^*)$ is a Hilbert space.

Proof. This follows from Assumption 2.31 and the completeness of $\mathcal{W}(V_0, V_0^*)$. \square

We are also able to specify initial conditions of solutions to PDEs via the following lemma, which is an easy consequence of the continuity of the embedding $\mathcal{W}(V_0, V_0^*) \subset C^0([0, T]; H_0)$.

Lemma 2.34. The embedding $W(V, V^*) \subset C_H^0$ holds, hence for every $u \in W(V, V^*)$ the evaluation $t \mapsto u(t)$ is well-defined for every $t \in [0, T]$. Furthermore, we have the inequality

$$\max_{t \in [0, T]} \|u(t)\|_{H(t)} \leq C \|u\|_{W(V, V^*)} \quad \forall u \in W(V, V^*).$$

In order to obtain a regularity result, we need to make the following natural assumption, which will also tell us that $W(V, H)$ is a Hilbert space.

Assumption 2.35. We assume that there is an evolving space equivalence between $W(V, H)$ and $\mathcal{W}(V_0, H_0)$.

Let us note that this assumption follows if, for example, the maps $\hat{S}(t)$ and $\hat{D}(t)$ of Theorem 2.32 satisfy $\hat{S}(t): H_0 \rightarrow H_0$ and $\hat{D}(t): V_0 \rightarrow H_0$, with both maps and $\hat{S}(t)^{-1}$ being bounded independently of $t \in [0, T]$, and if $\hat{S}(\cdot)u'(\cdot), \hat{D}(\cdot)u(\cdot) \in L^2(0, T; H_0)$ for $u \in \mathcal{W}(V_0, H_0)$.

Some density results With the help of the density result in Lemma 2.1, it is easy to prove the following lemma.

Lemma 2.36. The space $\mathcal{D}_V[0, T]$ is dense in $W(V, V^*)$.

The next few results are necessary to prove Lemma 3.5, which turns out to be vital for one of our existence proofs.

Lemma 2.37. For every $\eta \in \mathcal{D}_V(0, T)$, there exists a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_V(0, T)$ of the form

$$\eta_n(t) = \sum_{j=1}^n \zeta_j(t) \phi_t w_j \quad \text{where } \zeta_j \in \mathcal{D}(0, T) \text{ and } w_j \in V_0,$$

such that $\eta_n \rightarrow \eta$ in $W(V, V^*)$.

Proof. It suffices to show that for every $\psi \in \mathcal{D}((0, T); V_0)$, there exists a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset \mathcal{D}((0, T); V_0)$ of the form

$$\psi_n(t) = \sum_{j=1}^n \zeta_j(t) w_j \quad \text{where } \zeta_j \in \mathcal{D}(0, T) \text{ and } w_j \in V_0,$$

such that $\psi_n \rightarrow \psi$ in $\mathcal{W}(V_0, V_0^*)$.

Let w_j be an orthonormal basis for V_0 . Given $\psi \in \mathcal{D}((0, T); V_0)$, define

$$\psi_n(t) = \sum_{j=1}^n (\psi(t), w_j)_{V_0} w_j,$$

i.e., $\zeta_j(t) = (\psi(t), w_j)_{V_0}$. It is clear that ζ_j vanishes at the boundary (since ψ does), and $\zeta_j^{(m)}(t) = (\psi^{(m)}(t), w_j)_{V_0}$ also implies that $\zeta_j \in \mathcal{D}(0, T)$.

It remains to be checked that $\psi_n \rightarrow \psi$ in $\mathcal{W}(V_0, V_0^*)$. We have the pointwise convergence $\psi_n(t) \rightarrow \psi(t)$ in V_0 because w_j is a basis, and there is also the uniform bound $\|\psi_n(t)\|_{V_0} \leq \|\psi(t)\|_{V_0}$. So by the dominated convergence theorem,

$$\psi_n \rightarrow \psi \quad \text{in } L^2(0, T; V_0).$$

The same reasoning applied to ψ'_n allows us to conclude. □

Transport theorem Like in part (3) of Lemma 2.1, we want to differentiate the inner product on $H(t)$. Writing Lemma 2.27 in different notation, we obtain for $u, v \in C^1_H$ the transport theorem for C^1_H functions:

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = (\dot{u}(t), v(t))_{H(t)} + (u(t), \dot{v}(t))_{H(t)} + c(t; u(t), v(t)).$$

We can obtain a formula for general functions $u, v \in W(V, V^*)$ by means of a density argument.

Theorem 2.38 (Transport theorem). For all $u, v \in W(V, V^*)$, the map

$$t \mapsto (u(t), v(t))_{H(t)}$$

is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \dot{u}(t), v(t) \rangle_{V^*(t), V(t)} + \langle \dot{v}(t), u(t) \rangle_{V^*(t), V(t)} + c(t; u(t), v(t))$$

for almost every $t \in [0, T]$.

Proof. Given $u \in W(V, V^*)$, by Lemma 2.36, there exists a sequence $u_m \in \mathcal{D}_V[0, T]$ converging to u in $W(V, V^*)$. That is,

$$\begin{aligned} u_m &\rightarrow u && \text{in } L^2_V \\ \dot{u}_m &\rightarrow \dot{u} && \text{in } L^2_{V^*}. \end{aligned}$$

By the transport theorem for C^1_H functions, the u_m satisfy

$$\frac{d}{dt} \|u_m(t)\|_{H(t)}^2 = 2\langle \dot{u}_m(t), u_m(t) \rangle_{V^*(t), V(t)} + c(t; u_m(t), u_m(t)). \quad (2.16)$$

(We rewrote the inner product on $H(t)$ as a duality pairing since $u_m(t) \in V(t)$ and $\dot{u}_m(t) \in H(t)$). The statement (2.16) written in terms of weak derivatives is that for any $\zeta \in \mathcal{D}(0, T)$, it holds that

$$-\int_0^T \|u_m(t)\|_{H(t)}^2 \zeta'(t) = \int_0^T (2\langle \dot{u}_m(t), u_m(t) \rangle_{V^*(t), V(t)} + c(t; u_m(t), u_m(t))) \zeta(t). \quad (2.17)$$

Now we must pass to the limit in this equation. For the left hand side, because $u_m \rightarrow u$ in L^2_H , we have by the reverse triangle inequality

$$\int_0^T | \|u_m(t)\|_{H(t)} - \|u(t)\|_{H(t)} |^2 \leq \int_0^T \|u_m(t) - u(t)\|_{H(t)}^2 \rightarrow 0,$$

i.e., $\|u_m(\cdot)\|_{H(\cdot)} \rightarrow \|u(\cdot)\|_{H(\cdot)}$ in $L^2(0, T)$, which implies that

$$\|u_m(\cdot)\|_{H(\cdot)}^2 \rightarrow \|u(\cdot)\|_{H(\cdot)}^2 \quad \text{in } L^1(0, T).$$

Clearly, the functional $F: L^1(0, T) \rightarrow \mathbb{R}$, defined

$$F(y) = \int_0^T y(t) \zeta'(t),$$

is an element of $L^1(0, T)^*$ because $\zeta'(t)$ is bounded. Therefore, we have convergence of the left hand side of (2.17):

$$-\int_0^T \|u_m(t)\|_{H(t)}^2 \zeta'(t) \rightarrow -\int_0^T \|u(t)\|_{H(t)}^2 \zeta'(t).$$

To deal with the terms on the right hand side of (2.17), we require the estimates

$$\begin{aligned} & |\langle \dot{u}_m(t), u_m(t) \rangle_{V^*(t), V(t)} - \langle \dot{u}(t), u(t) \rangle_{V^*(t), V(t)}| \\ & \leq \|\dot{u}_m(t)\|_{V^*(t)} \|u_m(t) - u(t)\|_{V(t)} + \|\dot{u}_m(t) - \dot{u}(t)\|_{V^*(t)} \|u(t)\|_{V(t)} \end{aligned}$$

and

$$\begin{aligned} & |c(t; u_m(t), u_m(t)) - c(t; u(t), u(t))| \\ & \leq C_1 \left(\|u_m(t)\|_{H(t)} \|u_m(t) - u(t)\|_{H(t)} + \|u_m(t) - u(t)\|_{H(t)} \|u(t)\|_{H(t)} \right). \end{aligned}$$

With these, it is easy to show that

$$\begin{aligned} & \left| \int_0^T (2\langle \dot{u}_m(t), u_m(t) \rangle_{V^*(t), V(t)} + c(t; u_m(t), u_m(t))) \zeta(t) \right. \\ & \quad \left. - \int_0^T (2\langle \dot{u}(t), u(t) \rangle_{V^*(t), V(t)} + c(t; u(t), u(t))) \zeta(t) \right| \rightarrow 0. \end{aligned}$$

In other words, as $m \rightarrow \infty$, the equation (2.17) becomes

$$- \int_0^T \|u(t)\|_{H(t)}^2 \zeta'(t) = \int_0^T (2\langle \dot{u}(t), u(t) \rangle_{V^*(t), V(t)} + c(t; u(t), u(t))) \zeta(t), \quad (2.18)$$

which is precisely the statement

$$\frac{d}{dt} \|u(t)\|_{H(t)}^2 = 2\langle \dot{u}(t), u(t) \rangle_{V^*(t), V(t)} + c(t; u(t), u(t))$$

in the sense of distributions. From this, it follows that

$$\frac{d}{dt} (u(t), v(t))_{H(t)} = \langle \dot{u}(t), v(t) \rangle_{V^*(t), V(t)} + \langle \dot{v}(t), u(t) \rangle_{V^*(t), V(t)} + c(t; u(t), v(t)) \quad (2.19)$$

holds in the weak sense. So we have shown the transport theorem in the weak sense. However, because the right hand side of the above is in $L^1(0, T)$ (since the right hand side of (2.18) holds for every $\zeta \in \mathcal{D}(0, T)$) and because $(u(t), v(t))_{H(t)} \in L^1(0, T)$, it follows that $(u(t), v(t))_{H(t)}$ is a.e. equal to an absolutely continuous function, with (classical) derivative a.e., and therefore (2.19) exists in the classical sense. \square

We shall use the following corollary frequently without referencing in future sections.

Corollary 2.39 (Formula of partial integration). For all $u, v \in W(V, V^*)$, the formula of partial integration

$$\begin{aligned} & (u(T), v(T))_{H(T)} - (u(0), v(0))_{H_0} \\ & = \int_0^T \langle \dot{u}(t), v(t) \rangle_{V^*(t), V(t)} + \langle \dot{v}(t), u(t) \rangle_{V^*(t), V(t)} + c(t; u(t), v(t)) dt \end{aligned}$$

holds.

3 Formulation of the problem and statement of results

3.1 Precise formulation of the PDE

Having built up the essential function spaces and results, we are now in a position to formulate PDEs on evolving spaces. We continue with the framework and notation of §2; we reiterate in particular Assumptions 2.17, 2.24, and 2.31 (which relate respectively to the compatibility of the evolving Hilbert triple, a well-defined material derivative, and the evolving space equivalence). We are interested in the existence and uniqueness of solutions to equations of the form

$$\begin{aligned} \mathcal{L}\dot{u} + \mathcal{A}u + \mathcal{C}u &= f & \text{in } L^2_{V^*} \\ u(0) &= u_0 \end{aligned} \quad (\mathbf{P})$$

where we identify

$$\begin{aligned} (\mathcal{L}\dot{u})(t) &= \mathcal{L}(t)\dot{u}(t) \\ (\mathcal{A}u)(t) &= \mathcal{A}(t)u(t) \\ (\mathcal{C}u)(t) &= \mathcal{C}(t)u(t), \end{aligned}$$

with $\mathcal{L}(t)$ and $\mathcal{A}(t)$ being linear operators that satisfy the minimal assumptions given below, and

$$\mathcal{C}(t): H(t) \rightarrow H^*(t) \quad \text{is defined by} \quad \langle \mathcal{C}(t)v, w \rangle_{H^*(t), H(t)} = c(t; v, w),$$

with $c(t; \cdot, \cdot)$ the bilinear form in the definition of the weak material derivative (Definition 2.25). Note that $\mathcal{C}(t)$ is symmetric in the sense that $\langle \mathcal{C}(t)v(t), w(t) \rangle_{H^*(t), H(t)} = \langle \mathcal{C}(t)w(t), v(t) \rangle_{H^*(t), H(t)}$.

Assumptions 3.1 (Assumptions on $\mathcal{L}(t)$). In the following, all constants C_i are positive and independent of $t \in [0, T]$.

We shall assume that for all $g \in L^2_{V^*}$,

$$\mathcal{L}g \in L^2_{V^*} \quad \text{and} \quad C_1 \|g\|_{L^2_{V^*}} \leq \|\mathcal{L}g\|_{L^2_{V^*}} \leq C_2 \|g\|_{L^2_{V^*}}, \quad (\mathbf{L1})$$

and we assume that the restrictions $\mathcal{L}(t)|_{H(t)}$, $\mathcal{L}(t)|_{V(t)}$ satisfy

$$\begin{aligned} \mathcal{L}(t)|_{H(t)} &: H(t) \rightarrow H(t) \\ \mathcal{L}(t)|_{V(t)} &: V(t) \rightarrow V(t), \end{aligned}$$

and we simply write $\mathcal{L}(t)$ for these restrictions. Furthermore, for almost every $t \in [0, T]$, we assume

$$\langle \mathcal{L}(t)g, v \rangle_{V^*(t), V(t)} = \langle g, \mathcal{L}(t)v \rangle_{V^*(t), V(t)} \quad \forall g \in V^*(t), \forall v \in V(t) \quad (\mathbf{L2})$$

$$\|\mathcal{L}(t)h\|_{H(t)} \leq C_3 \|h\|_{H(t)} \quad \forall h \in H(t) \quad (\mathbf{L3})$$

$$(\mathcal{L}(t)h, h)_{H(t)} \geq C_4 \|h\|_{H(t)}^2 \quad \forall h \in H(t) \quad (\mathbf{L4})$$

$$\mathcal{L}v \in L^2_V \quad \forall v \in L^2_V \quad (\mathbf{L5})$$

$$v \in W(V, V^*) \iff \mathcal{L}v \in W(V, V^*), \quad (\mathbf{L6})$$

and we suppose the existence of a map $\dot{\mathcal{L}}(t): V(t) \rightarrow V^*(t)$ (and we identify $(\dot{\mathcal{L}}v)(t) = \dot{\mathcal{L}}(t)v(t)$) satisfying

$$\partial^\bullet(\mathcal{L}v) = \dot{\mathcal{L}}v + \mathcal{L}\dot{v} \in L_{V^*}^2 \quad \forall v \in W(V, V^*) \quad (\text{L7})$$

$$\|\dot{\mathcal{L}}(t)v\|_{V^*(t)} \leq C_5 \|v\|_{H(t)} \quad \forall v \in V(t). \quad (\text{L8})$$

Assumptions 3.2 (Assumptions on $\mathcal{A}(t)$). Suppose that the map

$$t \mapsto \langle \mathcal{A}(t)v(t), w(t) \rangle_{V^*(t), V(t)} \quad \forall v, w \in L_V^2$$

is measurable, and that there exist positive constants C_1 , C_2 and C_3 independent of t such that the following holds for almost every $t \in [0, T]$:

$$\langle \mathcal{A}(t)v, v \rangle_{V^*(t), V(t)} \geq C_1 \|v\|_{V(t)}^2 - C_2 \|v\|_{H(t)}^2 \quad \forall v \in V(t) \quad (\text{A1})$$

$$|\langle \mathcal{A}(t)v, w \rangle_{V^*(t), V(t)}| \leq C_3 \|v\|_{V(t)} \|w\|_{V(t)} \quad \forall v, w \in V(t). \quad (\text{A2})$$

Observe that we have generalised the PDE (1.2) by introducing the operator \mathcal{L} . The standard equation

$$\dot{u} + \mathcal{A}u + \mathcal{C}u = f$$

is a special case of **(P)** when $\mathcal{L} = \text{Id}$. Let us mention that our demands in Assumptions 3.1 are (of course) automatically met in this case. Also, there is no loss of generality by considering the equation **(P)** instead of the more natural equation

$$\mathcal{L}\dot{u} + \mathcal{A}u = f.$$

The results we state below still hold for this case. We include the operator \mathcal{C} purely because it is convenient for some of our applications in §6.

Implicit in **(P)** is the claim that $\mathcal{A}u$ and $\mathcal{C}u$ are elements of $L_{V^*}^2$. The fact $\mathcal{A}u \in L_{V^*}^2$ follows by the weak (and thus strong) measurability of $t \mapsto \phi_t^* \mathcal{A}(t)u(t)$ and the boundedness of $\mathcal{A}(t)$, and similarly one obtains the result $\mathcal{C}u \in L_{V^*}^2$.

Remark 3.3. We showed in Lemma 2.34 that specifying the initial condition as in the equation **(P)** is well-defined.

Let us mention an important consequence of the transport theorem (Theorem 2.38) and assumptions (L6) and (L7).

Lemma 3.4. For every $v, w \in W(V, V^*)$, the map $t \mapsto (\mathcal{L}(t)v(t), w(t))_{H(t)}$ is classically differentiable almost everywhere with

$$\begin{aligned} \frac{d}{dt}(\mathcal{L}(t)v(t), w(t))_{H(t)} &= \langle \mathcal{L}(t)\dot{v}(t), w(t) \rangle_{V^*(t), V(t)} + \langle \mathcal{L}(t)\dot{w}(t), v(t) \rangle_{V^*(t), V(t)} \\ &\quad + \langle \mathcal{M}(t)v(t), w(t) \rangle_{V^*(t), V(t)} \end{aligned} \quad (\text{L9})$$

where $\mathcal{M}(t): V(t) \rightarrow V^*(t)$ is the operator

$$\langle \mathcal{M}(t)v, w \rangle_{V^*(t), V(t)} := \langle \dot{\mathcal{L}}(t)v, w \rangle_{V^*(t), V(t)} + \langle \mathcal{C}(t)\mathcal{L}(t)v, w \rangle_{V^*(t), V(t)}$$

which generates the bounded bilinear form $m(t; \cdot, \cdot): V(t) \times V(t) \rightarrow \mathbb{R}$:

$$m(t; v, w) = \langle \mathcal{M}(t)v, w \rangle_{V^*(t), V(t)}.$$

To conclude this preliminary subsection we state and prove the following lemma which is used in §5.3.

Lemma 3.5. Let $u \in L_V^2$ and $g \in L_{V^*}^2$. Then

$$\dot{u} \text{ exists and } \mathcal{L}\dot{u} = g$$

if and only if

$$\frac{d}{dt}(\mathcal{L}(t)u(t), \phi_t v_0)_{H(t)} = \langle g(t) + \mathcal{M}(t)u(t), \phi_t v_0 \rangle_{V^*(t), V(t)} \text{ for all } v_0 \in V_0 \quad (3.1)$$

in the weak sense.

Proof of Lemma 3.5. Suppose first that $\mathcal{L}\dot{u} = g$, that is, $\partial^\bullet(\mathcal{L}u) = \dot{\mathcal{L}}u + g$ (note that this is sensible because $u \in W(V, V^*)$, and so $\mathcal{L}u \in W(V, V^*)$ by (L6)). This means

$$\int_0^T (\mathcal{L}(t)u(t), \dot{\eta}(t))_{H(t)} = - \int_0^T \langle g(t) + \dot{\mathcal{L}}(t)u(t), \eta(t) \rangle_{V^*(t), V(t)} - \int_0^T c(t; \mathcal{L}(t)u(t), \eta(t))$$

holds for all $\eta \in \mathcal{D}_V$. Picking $\eta(t) = \zeta(t)\phi_t v_0$, where $\zeta \in \mathcal{D}(0, T)$ and $v_0 \in V_0$, we obtain

$$\begin{aligned} \int_0^T \zeta'(t)(\mathcal{L}(t)u(t), \phi_t v_0)_{H(t)} &= - \int_0^T \zeta(t) \langle g(t) + \dot{\mathcal{L}}(t)u(t), \phi_t v_0 \rangle_{V^*(t), V(t)} \\ &\quad - \int_0^T \zeta(t) c(t; \mathcal{L}(t)u(t), \phi_t v_0) \\ &= - \int_0^T \zeta(t) \langle g(t) + \mathcal{M}(t)u(t), \phi_t v_0 \rangle_{V^*(t), V(t)}, \end{aligned}$$

which is exactly (3.1).

For the converse, first, we see from Lemma 2.37 that given any $\eta \in \mathcal{D}_V(0, T)$, there exist functions $\eta_n \in \mathcal{D}_V(0, T)$ of the form

$$\eta_n(t) = \sum_j \zeta_j(t) \phi_t w_j$$

with $\zeta_j \in \mathcal{D}(0, T)$ and $w_j \in V_0$ such that $\|\eta - \eta_n\|_{W(V, V^*)} \rightarrow 0$. Now, (3.1) states that

$$\int_0^T (\mathcal{L}(t)u(t), \zeta'(t)\phi_t v_0)_{H(t)} = - \int_0^T \langle g(t) + \mathcal{M}(t)u(t), \zeta(t)\phi_t v_0 \rangle_{V^*(t), V(t)}$$

holds for all $\zeta \in \mathcal{D}(0, T)$ and all $v_0 \in V_0$. In particular, we may pick $\zeta = \zeta_j$ and $v_0 = w_j$ and sum up over j to obtain

$$\int_0^T (\mathcal{L}(t)u(t), \dot{\eta}_n(t))_{H(t)} = - \int_0^T \langle g(t) + \mathcal{M}(t)u(t), \eta_n(t) \rangle_{V^*(t), V(t)}.$$

Passing to the limit and using the convergence above, we find

$$\begin{aligned} \int_0^T (\mathcal{L}(t)u(t), \dot{\eta}(t))_{H(t)} &= - \int_0^T \langle g(t) + \mathcal{M}(t)u(t), \eta(t) \rangle_{V^*(t), V(t)} \\ &= - \int_0^T \langle g(t) + \dot{\mathcal{L}}(t)u(t) + \mathcal{C}(t)\mathcal{L}(t)u(t), \eta(t) \rangle_{V^*(t), V(t)} \end{aligned}$$

for arbitrary $\eta \in \mathcal{D}_V(0, T)$, i.e., we have the existence of $\partial^\bullet(\mathcal{L}u) = g + \dot{\mathcal{L}}u$ which, thanks to assumptions (L6) and (L7), implies that $\mathcal{L}\dot{u} = g$. \square

3.2 Well-posedness and regularity

We begin with a well-posedness theorem which is proved in §4.

Theorem 3.6 (Well-posedness of **(P)**). Under the assumptions in Assumptions 3.1 and 3.2, for $f \in L^2_{V^*}$ and $u_0 \in H_0$, there is a unique solution $u \in W(V, V^*)$ satisfying **(P)** such that

$$\|u\|_{W(V, V^*)} \leq C \left(\|u_0\|_{H_0} + \|f\|_{L^2_{V^*}} \right).$$

A sketch of a second proof of the theorem will be presented in §5.3 where we utilise a Galerkin method.

Now, suppose we now know that $f \in L^2_H$ and $u_0 \in V_0$. Can we expect the same regularity on the solution u as holds in the case of stationary spaces? It turns out that we can obtain $\dot{u} \in L^2_H$ under some additional assumptions on the differentiability of $\mathcal{A}(t)$.

Before we list these assumptions, let us just note that if we define bilinear forms $l(t; \cdot, \cdot): V^*(t) \times V(t) \rightarrow \mathbb{R}$ and $a(t; \cdot, \cdot): V(t) \times V(t) \rightarrow \mathbb{R}$ to satisfy

$$\begin{aligned} l(t; g, w) &= \langle \mathcal{L}(t)g, w \rangle_{V^*(t), V(t)} \\ a(t; v, w) &= \langle \mathcal{A}(t)v, w \rangle_{V^*(t), V(t)}, \end{aligned}$$

then the problem **(P)** is in fact equivalent to

$$\begin{aligned} l(t; \dot{u}(t), v(t)) + a(t; u(t), v(t)) + c(t; u(t), v(t)) &= \langle f(t), v(t) \rangle_{V^*(t), V(t)} \\ u(0) &= u_0 \end{aligned} \tag{P'}$$

for all $v \in L^2_V$ and for almost every $t \in [0, T]$. It is this form of the problem that turns out to be more convenient to work with to show regularity. To see one side of the equivalence, we can take the duality pairing of **(P)** with $v \in L^2_V$ where $v(t) = \zeta(t)\phi_t v_0$ and $\zeta \in \mathcal{D}(0, T)$ and $v_0 \in V_0$ to give **(P')**. The reverse implication is trivial.

Definition 3.7. We define the space

$$\tilde{C}_V^1 = \{u \in C_V^0 \mid \dot{u}(t) \text{ exists for almost every } t \in [0, T]\}.$$

Assumptions 3.8 (Further assumptions on $a(t; \cdot, \cdot)$). Suppose that $a(t; \cdot, \cdot)$ has the form

$$a(t; \cdot, \cdot) = a_s(t; \cdot, \cdot) + a_n(t; \cdot, \cdot)$$

where

$$a_s(t; \cdot, \cdot): V(t) \times V(t) \rightarrow \mathbb{R}$$

$$a_n(t; \cdot, \cdot): V(t) \times H(t) \rightarrow \mathbb{R}$$

are bilinear forms (we allow the possibility $a_n \equiv 0$). Suppose that there exist positive constants C_1 , C_2 and C_3 independent of t such that for almost every $t \in [0, T]$,

$$|a_n(t; v, w)| \leq C_1 \|v\|_{V(t)} \|w\|_{H(t)} \quad \forall v \in V(t), w \in H(t) \quad (\text{A3})$$

$$|a_s(t; v, w)| \leq C_2 \|v\|_{V(t)} \|w\|_{V(t)} \quad \forall v, w \in V(t) \quad (\text{A4})$$

$$a_s(t; v, v) \geq 0 \quad \forall v \in V(t) \quad (\text{A5})$$

$$\frac{d}{dt} a_s(t; y(t), y(t)) = 2a_s(t; y(t), \dot{y}(t)) + r(t; y(t)) \quad \forall y \in \tilde{C}_V^1, \quad (\text{A6})$$

where the $\frac{d}{dt}$ here is the classical derivative, and $r(t; \cdot): V(t) \rightarrow \mathbb{R}$ satisfies

$$|r(t; v)| \leq C_3 \|v\|_{V(t)}^2 \quad \forall v \in V(t). \quad (\text{A7})$$

Remark 3.9. Note that we require only a part of the bilinear form $a(t; \cdot, \cdot)$ to be differentiable; however, any potentially non-differentiable terms require the stronger boundedness condition (A3).

As alluded to above, it is permissible to take $a_n \equiv 0$ so that $a \equiv a_s$. In this case, we are in the same situation as in Assumptions 3.2 except with the addition of (A6) and (A7).

We have the following regularity result proved in §4 under appropriate assumptions on the data.

Theorem 3.10 (Regularity of the solution to **(P)**). Under the assumptions in Assumptions 3.1, 3.2 and 3.8, for $f \in L_H^2$ and $u_0 \in V_0$, there is a unique solution $u \in W(V, H)$ satisfying **(P)** such that

$$\|u\|_{W(V, H)} \leq C \left(\|u_0\|_{V_0} + \|f\|_{L_H^2} \right).$$

4 Proof of well-posedness

We use a generalisation of the Lax–Milgram theorem sometimes called the Banach–Nečas–Babuška theorem to establish existence. See [17, §2.1.3] for its proof.

Theorem 4.1 (Banach–Nečas–Babuška). Let X be a Banach space and let Y be a reflexive Banach space. Suppose $d(\cdot, \cdot): X \times Y \rightarrow \mathbb{R}$ is a bounded bilinear form and $f \in Y^*$. Then there is a unique solution $x \in X$ to the problem

$$d(x, y) = \langle f, y \rangle_{Y^*, Y} \quad \text{for all } y \in Y$$

satisfying

$$\|x\|_X \leq C \|f\|_{Y^*} \quad (4.1)$$

if and only if

1. There exists $\alpha > 0$ such that

$$\inf_{x \in X} \sup_{y \in Y} \frac{d(x, y)}{\|x\|_X \|y\|_Y} \geq \alpha. \quad (\text{“inf-sup condition”})$$

2. For arbitrary $y \in Y$, if

$$d(x, y) = 0 \text{ holds for all } x \in X,$$

then $y = 0$.

Moreover, the estimate (4.1) holds with the constant $C = \frac{1}{\alpha}$.

Consider the equation **(P)**:

$$\begin{aligned} \mathcal{L}\dot{u} + \mathcal{A}u + \mathcal{C}u &= f & \text{in } L^2_{V^*} \\ u(0) &= u_0 \end{aligned}$$

where $f \in L^2_{V^*}$ and $u_0 \in H_0$. By considering a suitable initial value problem on a fixed domain we know that there is a function $y \in \mathcal{W}(V_0, V_0^*)$ with $y(0) = u_0$ and

$$\|y\|_{W(V_0, V_0^*)} \leq C \|u_0\|_{H_0}.$$

Then the function $\tilde{y}(\cdot) = \phi(\cdot)y(\cdot)$ is such that $\tilde{y} \in W(V, V^*)$ with $\tilde{y}(0) = u_0$. So then we can transform **(P)** into a PDE with zero initial condition if we set $w = u - \tilde{y}$:

$$\begin{aligned} \mathcal{L}\dot{w} + \mathcal{A}w + \mathcal{C}w &= \tilde{f} \\ w(0) &= 0 \end{aligned} \quad (\mathbf{P}_0)$$

where $\tilde{f} := f - \mathcal{L}\partial^\bullet \tilde{y} - \mathcal{A}\tilde{y} - \mathcal{C}\tilde{y} \in L^2_{V^*}$. It is clear that well-posedness of **(P₀)** translates into well-posedness of **(P)**. The idea is to apply the Banach–Nečas–Babuška theorem to the problem **(P₀)** with $X = W_0(V, V^*)$, $Y = L^2_V$, and the bilinear form

$$d(u, v) = \langle \mathcal{L}\dot{u}, v \rangle_{L^2_{V^*}, L^2_V} + \langle \mathcal{A}u, v \rangle_{L^2_{V^*}, L^2_V} + \langle \mathcal{C}u, v \rangle_{L^2_{V^*}, L^2_V}.$$

Remark 4.2. The space $W_0(V, V^*)$ is indeed a Hilbert space because by Lemma 2.34, it is a closed linear subspace of $W(V, V^*)$.

The arguments in the next two lemmas follow §4 in [27]. See also [17, §6.1.2].

Lemma 4.3. For all $w \in W_0(V, V^*)$, there exists a function $v_w \in L_V^2$ such that

$$\langle \mathcal{L}\dot{w}, v_w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{A}w, v_w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, v_w \rangle_{L_{V^*}^2, L_V^2} \geq C \|w\|_{W(V, V^*)} \|v_w\|_{L_V^2}.$$

Proof. This proof requires two estimates.

First estimate Let $w \in W_0(V, V^*)$ and set $w_\gamma(t) = e^{-\gamma t}w(t)$. Note that $w_\gamma \in W_0(V, V^*)$ too with $\dot{w}_\gamma(t) = e^{-\gamma t}\dot{w}(t) - \gamma w_\gamma(t)$, so

$$\langle \mathcal{L}(t)\dot{w}_\gamma(t), w(t) \rangle_{V^*(t), V(t)} = \langle \mathcal{L}(t)\dot{w}(t) - \gamma \mathcal{L}(t)w(t), w_\gamma(t) \rangle_{V^*(t), V(t)}.$$

Rearranging, integrating, and then using (L9):

$$\begin{aligned} \langle \mathcal{L}\dot{w}, w_\gamma \rangle_{L_{V^*}^2, L_V^2} &= \frac{1}{2} \left(\langle \mathcal{L}\dot{w}, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{L}\dot{w}_\gamma, w \rangle_{L_{V^*}^2, L_V^2} \right) + \frac{1}{2} \gamma \langle \mathcal{L}w, w_\gamma \rangle_{L_H^2} \\ &= \frac{1}{2} \int_0^T \frac{d}{dt} \langle \mathcal{L}(t)w(t), w_\gamma(t) \rangle_{H(t)} - \frac{1}{2} \langle \mathcal{M}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2} \gamma \langle \mathcal{L}w, w_\gamma \rangle_{L_H^2} \\ &\geq -\frac{1}{2} \langle \mathcal{M}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2} \gamma \langle \mathcal{L}w, w_\gamma \rangle_{L_H^2} \end{aligned} \quad (4.2)$$

as $\langle \mathcal{L}(T)w(T), w_\gamma(T) \rangle_{H(T)} = e^{-\gamma T} \langle \mathcal{L}(T)w(T), w(T) \rangle_{H(T)} \geq 0$ by (L4). Hence

$$\begin{aligned} &\langle \mathcal{L}\dot{w}, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{A}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} \\ &\geq \langle \mathcal{A}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} - \frac{1}{2} \langle \mathcal{M}w, w_\gamma \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2} \gamma \langle \mathcal{L}w, w_\gamma \rangle_{L_H^2} \\ &\geq \int_0^T e^{-\gamma t} \left(C_1 \|w(t)\|_{V(t)}^2 - C_2 \|w(t)\|_{H(t)}^2 \right) - \frac{1}{2} \int_0^T C_3 e^{-\gamma t} \|w(t)\|_{H(t)}^2 \\ &\quad + \frac{\gamma C_4}{2} \int_0^T e^{-\gamma t} \|w(t)\|_{H(t)}^2 \\ &\quad \text{(by the coercivity of } \mathcal{A}(t) \text{ and } \mathcal{L}(t) \text{ and the boundedness of } \mathcal{C}(t) \text{ and } \mathcal{M}(t)) \\ &= C_1 \int_0^T e^{-\gamma t} \|w(t)\|_{V(t)}^2 + \frac{\gamma C_4 - C_3 - 2C_2}{2} \int_0^T e^{-\gamma t} \|w(t)\|_{H(t)}^2 \\ &\geq e^{-\gamma T} C_1 \|w\|_{L_V^2}^2 \end{aligned} \quad (E1)$$

with the final inequality holding if we choose γ such that $\gamma C_4 > C_3 + 2C_2$.

Second estimate Now, by the Riesz representation theorem, there exists $z \in L_V^2$ such that

$$\langle \mathcal{L}\dot{w}, v \rangle_{L_{V^*}^2, L_V^2} = (z, v)_{L_V^2} \quad \text{for all } v \in L_V^2 \quad (4.3)$$

with $\|z\|_{L_V^2} = \|\mathcal{L}\dot{w}\|_{L_{V^*}^2}$. We have

$$\begin{aligned} \langle \mathcal{L}\dot{w}, z \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{A}w, z \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, z \rangle_{L_{V^*}^2, L_V^2} &\geq \|z\|_{L_V^2}^2 - \int_0^T C_5 \|w(t)\|_{V(t)} \|z(t)\|_{V(t)} \\ &\quad \text{(by (4.3) and the bounds on } a(t; \cdot, \cdot) \text{ and } c(t; \cdot, \cdot)) \\ &\geq C_6 \|z\|_{L_V^2}^2 - C_7 \|w\|_{L_V^2}^2 \\ &\quad \text{(using Young's inequality with } \epsilon > 0 \text{ chosen small enough)} \\ &= C_6 \|\mathcal{L}\dot{w}\|_{L_{V^*}^2}^2 - C_7 \|w\|_{L_V^2}^2. \end{aligned} \quad (\text{E2})$$

Combining the estimates Estimate (E2) gives us control of $\mathcal{L}\dot{w}$ at the expense of w , but the latter is controlled by estimate (E1). So let us put $v_w := z + \mu w_\gamma$ where $\mu > 0$ is a constant to be determined and consider:

$$\begin{aligned} \langle \mathcal{L}\dot{w}, v_w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{A}w, v_w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, v_w \rangle_{L_{V^*}^2, L_V^2} \\ \geq C_6 \|\mathcal{L}\dot{w}\|_{L_{V^*}^2}^2 - C_7 \|w\|_{L_V^2}^2 + \mu e^{-\gamma T} C_1 \|w\|_{L_V^2}^2 \\ \geq C_6 \|\mathcal{L}\dot{w}\|_{L_{V^*}^2}^2 + C_8 \|w\|_{L_V^2}^2 \quad \text{(if } \mu \text{ is large enough)} \\ \geq C_9 \|w\|_{W(V, V^*)}^2 \end{aligned}$$

thanks to (L1). Finally, because

$$\begin{aligned} \|v_w\|_{L_V^2} &\leq \|z\|_{L_V^2} + \mu \|w_\gamma\|_{L_V^2} \\ &= \|\mathcal{L}\dot{w}\|_{L_{V^*}^2} + \mu \left(\int_0^T |e^{-\gamma t}|^2 \|w(t)\|_{V(t)}^2 \right)^{\frac{1}{2}} \\ &\leq \|\mathcal{L}\dot{w}\|_{L_{V^*}^2} + \mu \|w\|_{L_V^2} \\ &\leq C_{10} \|w\|_{W(V, V^*)} \end{aligned} \quad (\text{by (L1)})$$

we end up with

$$\langle \mathcal{L}\dot{w}, v_w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{A}w, v_w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, v_w \rangle_{L_{V^*}^2, L_V^2} \geq C \|w\|_{W(V, V^*)} \|v_w\|_{L_V^2} \square$$

Lemma 4.4. If given arbitrary $v \in L_V^2$, the equality

$$\langle \mathcal{L}\dot{w}, v \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{A}w, v \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, v \rangle_{L_{V^*}^2, L_V^2} = 0 \quad (4.4)$$

holds for all $w \in W_0(V, V^*)$, then necessarily $v = 0$.

Proof. Define the operator $\tilde{\mathcal{A}}(t) : V(t) \rightarrow V^*(t)$ by

$$\langle \tilde{\mathcal{A}}(t)v(t), \eta(t) \rangle_{V^*(t), V(t)} := \langle \mathcal{A}(t)\eta(t), v(t) \rangle_{V^*(t), V(t)}$$

and identify $(\tilde{\mathcal{A}}v)(t) = \tilde{\mathcal{A}}(t)v(t)$. Take $w = \eta \in \mathcal{D}_V$ in (4.4) and rearrange to give

$$\begin{aligned} \langle \mathcal{L}\dot{\eta}, v \rangle_{L_H^2} &= \langle \mathcal{L}v, \dot{\eta} \rangle_{L_H^2} = -\langle \tilde{\mathcal{A}}v, \eta \rangle_{L_{V^*}^2, L_V^2} - \langle \mathcal{C}v, \eta \rangle_{L_{V^*}^2, L_V^2} \\ &= -\langle \tilde{\mathcal{A}}v - \mathcal{C}\mathcal{L}v + \mathcal{C}v, \eta \rangle_{L_{V^*}^2, L_V^2} - \langle \mathcal{C}\mathcal{L}v, \eta \rangle_{L_{V^*}^2, L_V^2} \end{aligned}$$

where we used (L2), the symmetric property of $\mathcal{L}(t)$. (We could not simply have used \mathcal{A} in place $\tilde{\mathcal{A}}$ above because $a(t; \cdot, \cdot)$ may not be symmetric.) This tells us that $\partial^\bullet(\mathcal{L}v) = \tilde{\mathcal{A}}v - \mathcal{C}\mathcal{L}v + \mathcal{C}v \in L_{V^*}^2$, and so $\mathcal{L}v \in W(V, V^*)$ (we already have $\mathcal{L}v \in L_V^2$ from (L5)). So

$$\langle \partial^\bullet(\mathcal{L}v), \eta \rangle_{L_{V^*}^2, L_V^2} = \langle (\tilde{\mathcal{A}} - \mathcal{C}\mathcal{L} + \mathcal{C})v, \eta \rangle_{L_{V^*}^2, L_V^2} \quad \forall \eta \in \mathcal{D}_V.$$

By the density of $\mathcal{D}((0, T); V_0) \subset L^2(0, T; V_0)$, we have the density of $\mathcal{D}_V \subset L_V^2$, which implies

$$\langle \partial^\bullet(\mathcal{L}v), w \rangle_{L_{V^*}^2, L_V^2} = \langle (\tilde{\mathcal{A}} - \mathcal{C}\mathcal{L} + \mathcal{C})v, w \rangle_{L_{V^*}^2, L_V^2} \quad \forall w \in L_V^2. \quad (4.5)$$

If in particular $w \in W_0(V, V^*)$, then we can use (4.4) on the right hand side of (4.5) to give

$$\langle \mathcal{L}\dot{w}, v \rangle_{L_{V^*}^2, L_V^2} + \langle \partial^\bullet(\mathcal{L}v), w \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{C}w, \mathcal{L}v \rangle_{L_{V^*}^2, L_V^2} = 0 \quad \forall w \in W_0(V, V^*). \quad (4.6)$$

Using (L2), $(\mathcal{L}(t)w(t), v(t))_{H(t)} = (\mathcal{L}(t)v(t), w(t))_{H(t)}$, and so

$$\begin{aligned} \frac{d}{dt}(\mathcal{L}(t)w(t), v(t))_{H(t)} &= \langle \partial^\bullet(\mathcal{L}(t)v(t)), w(t) \rangle_{V^*(t), V(t)} + \langle \dot{w}(t), \mathcal{L}(t)v(t) \rangle_{V^*(t), V(t)} \\ &\quad + \langle \mathcal{C}(t)w(t), \mathcal{L}(t)v(t) \rangle_{H^*(t), H(t)} \end{aligned}$$

to which another application of (L2) shows us that (4.6) is exactly

$$\int_0^T \frac{d}{dt}(\mathcal{L}(t)w(t), v(t))_{H(t)} = (\mathcal{L}(T)w(T), v(T))_{H(T)} = 0$$

for all $w \in W_0(V, V^*)$. Thus we have shown that $v(T) = 0$.

Let $0 > \gamma \in \mathbb{R}$ and set $w(t) = v_\gamma(t) = e^{-\gamma t}v(t)$ in (4.5) to obtain

$$0 = \langle \partial^\bullet(\mathcal{L}v), v_\gamma \rangle_{L_{V^*}^2, L_V^2} - \langle (\tilde{\mathcal{A}} - \mathcal{C}\mathcal{L} + \mathcal{C})v, v_\gamma \rangle_{L_{V^*}^2, L_V^2}. \quad (4.7)$$

We showed that $\mathcal{L}v \in W(V, V^*)$ earlier; by (L6), $v \in W(V, V^*)$ too, and so we can apply (L7) to the first term on the right hand side of (4.7):

$$\begin{aligned} \langle \partial^\bullet(\mathcal{L}v), v_\gamma \rangle_{L_{V^*}^2, L_V^2} &= \langle \dot{\mathcal{L}}v, v_\gamma \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{L}\dot{v}, v_\gamma \rangle_{L_{V^*}^2, L_V^2} \\ &= \langle \dot{\mathcal{L}}v, v_\gamma \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2} \left(\langle \mathcal{L}\dot{v}, v_\gamma \rangle_{L_{V^*}^2, L_V^2} + \langle \mathcal{L}\dot{v}_\gamma, v \rangle_{L_{V^*}^2, L_V^2} \right) + \frac{1}{2}\gamma \langle \mathcal{L}v, v_\gamma \rangle_{L_H^2} \\ &\quad \text{(follows like the equation (4.2))} \\ &\leq \frac{1}{2} \langle \dot{\mathcal{L}}v, v_\gamma \rangle_{L_{V^*}^2, L_V^2} - \frac{1}{2} \langle \mathcal{C}v_\gamma, \mathcal{L}v \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2}\gamma \langle \mathcal{L}v, v_\gamma \rangle_{L_H^2} \\ &\quad \text{(since } v(T) = 0 \text{ and by coercivity of } \mathcal{L}(0)) \end{aligned}$$

Note that (L8) together with Young's inequality implies

$$\begin{aligned} \langle \dot{\mathcal{L}}(t)v(t), v(t) \rangle_{V^*(t), V(t)} &\leq \|\dot{\mathcal{L}}(t)v(t)\|_{V^*(t)} \|v(t)\|_{V(t)} \leq C_5 \|v(t)\|_{H(t)} \|v(t)\|_{V(t)} \\ &\leq C_\epsilon \|v(t)\|_{H(t)}^2 + \epsilon \|v(t)\|_{V(t)}^2. \end{aligned}$$

Therefore, (4.7) becomes

$$\begin{aligned} 0 &\leq \frac{1}{2} \langle \dot{\mathcal{L}}v, v_\gamma \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2} \langle \mathcal{C}v_\gamma, \mathcal{L}v \rangle_{L_{V^*}^2, L_V^2} + \frac{1}{2} \gamma \langle \mathcal{L}v, v_\gamma \rangle_{L_H^2} - \langle (\tilde{\mathcal{A}} + \mathcal{C})v, v_\gamma \rangle_{L_{V^*}^2, L_V^2} \\ &= \frac{1}{2} \int_0^T e^{-\gamma t} \langle \dot{\mathcal{L}}(t)v(t), v(t) \rangle_{V^*(t), V(t)} + \frac{1}{2} \int_0^T e^{-\gamma t} c(t; \mathcal{L}(t)v(t), v(t)) \\ &\quad + \frac{1}{2} \int_0^T \gamma e^{-\gamma t} \langle \mathcal{L}(t)v(t), v(t) \rangle_{H(t)} - \int_0^T e^{-\gamma t} \langle \tilde{\mathcal{A}}(t)v(t) + \mathcal{C}(t)v(t), v(t) \rangle_{V^*(t), V(t)} \\ &\leq (C_1 + \gamma C_2) \int_0^T e^{-\gamma t} \|v(t)\|_{H(t)}^2 - C_a \int_0^T e^{-\gamma t} \|v(t)\|_{V(t)}^2 \end{aligned}$$

using the assumptions (L3) and (L8) (boundedness) and (L4) and (A1) (coercivity). If we pick $\gamma = -\frac{C_1}{C_2}$, it follows that $v = 0$ in L_V^2 . \square

Proof of Theorem 3.6. The inf-sup condition (which is an easy consequence of Lemma 4.3) in combination with Lemma 4.4 furnishes the requirements of the Banach–Nečas–Babuška theorem (Theorem 4.1) thus yielding the existence and uniqueness of a solution $w \in W_0(V, V^*)$ to

$$\begin{aligned} \mathcal{L}w + \mathcal{A}w + \mathcal{C}w &= \tilde{f} \\ w(0) &= 0 \end{aligned}$$

where $\tilde{f} \in L_{V^*}^2$ is arbitrary. Hence, we have well-posedness of (\mathbf{P}_0) with the estimate

$$\|w\|_{W(V, V^*)} \leq C \|\tilde{f}\|_{L_{V^*}^2}.$$

From this well-posedness result, we also obtain unique solvability of (\mathbf{P}) by setting $u = w + \tilde{y}$ (note that w depends on \tilde{y}), with the solution $u \in W(V, V^*)$ satisfying

$$\|u\|_{W(V, V^*)} \leq C \left(\|f\|_{L_{V^*}^2} + \|u_0\|_{H_0} \right). \quad \square$$

5 Galerkin approximation

In this section we abstract the pushed-forward Galerkin method used in [11] for the advection-diffusion equation on an evolving hypersurface.

5.1 Finite-dimensional spaces

We start by supposing that $\{\chi_j^0\}_{j \in \mathbb{N}}$ is a basis of V_0 and H_0 . We can turn this into a basis of $V(t)$ and $H(t)$ with the help of the continuous map ϕ_t .

Lemma 5.1. With $\chi_j^t := \phi_t(\chi_j^0)$ for each $j \in \mathbb{N}$, the set $\{\chi_j^t\}_{j \in \mathbb{N}}$ is a countable basis of $H(t)$ and $V(t)$.

The next result is an extremely useful property of the basis functions following from Remark 2.21 (see [11] for the finite element analogue).

Lemma 5.2 (Transport property of basis functions). The basis $\{\chi_j^t\}_{j \in \mathbb{N}}$ satisfies the transport property

$$\dot{\chi}_j^t = 0.$$

We now construct the approximation spaces in which the discrete solutions lie.

Definition 5.3 (Approximation spaces). For each $N \in \mathbb{N}$ and each $t \in [0, T]$, define

$$V_N(t) = \text{span}\{\chi_1^t, \dots, \chi_N^t\} \subset V(t).$$

Clearly $V_N(t) \subset V_{N+1}(t)$ and $\bigcup_{j \in \mathbb{N}} V_j(t)$ is dense in $V(t)$. Define

$$L_{V_N}^2 = \{u \in L_V^2 \mid u(t) = \sum_{j=1}^N \alpha_j(t) \chi_j^t \text{ where } \alpha_j: [0, T] \rightarrow \mathbb{R}\}.$$

Similarly, $L_{V_N}^2 \subset L_{V_{N+1}}^2$, and we shall state a density result below.

Lemma 5.4. The space $\bigcup_{j \in \mathbb{N}} L_{V_j}^2$ is dense in L_V^2 .

The proof of this lemma follows from the density of the embedding $\bigcup_{j \in \mathbb{N}} L^2(0, T; V_j(0)) \subset L^2(0, T; V_0)$ and from the fact that $L^2(0, T; V_j(0)) \subset L^2(0, T; V_{j+1}(0))$.

Remark 5.5. If $u \in L_{V_N}^2$ with $u(t) = \sum_{j=1}^N \alpha_j(t) \chi_j^t$ has coefficients $\alpha_j \in C^1([0, T])$, then $u \in C_V^1$ with strong material derivative $\dot{u}(t) = \sum_{j=1}^N \alpha_j'(t) \chi_j^t$, and $\dot{u} \in L_{V_N}^2$. Our Galerkin ansatz (see below) has coefficients in a slightly less convenient space.

Galerkin ansatz. Later on, we construct finite-dimensional solutions which have the form

$$u_N(t) = \sum_{j=1}^N u_j^N(t) \chi_j^t \in V_N(t)$$

where the $u_j^N: [0, T] \rightarrow \mathbb{R}$ turn out to be absolutely continuous coefficient functions with $\dot{u}_j^N \in L^2(0, T)$. It holds that $u_N \in L_V^2$ and by definition, $u_N \in L_{V_N}^2$.

Lemma 5.6. The material derivative of $u_N \in L^2_{V_N}$ is $\dot{u}_N \in L^2_{V_N}$ with $\dot{u}_N(t) = \sum_{j=1}^N \dot{u}_j^N(t) \chi_j^t$.

We skip the proof which is straightforward: just use the definition of the weak material derivative and perform some manipulations.

Remark 5.7. In the previous lemma, we could not have calculated the strong material derivative of u_N via the formula (2.7) because the pullback

$$\phi_{-(\cdot)} u_N(t) = \sum_{j=1}^n u_j^N(\cdot) \chi_j^0$$

is not necessarily in $C^1([0, T]; V_0)$ since the u_j^N are merely differentiable.

Definition 5.8 (Projection operators). For each $t \in [0, T]$, define a projection operator $P_N^t: H(t) \rightarrow V_N(t)$ by the formula

$$(P_N^t u - u, v_N)_{H(t)} = 0 \quad \text{for all } v_N \in V_N(t).$$

It follows that $(P_N^t)^2 = P_N^t$,

$$\|P_N^t u\|_{H(t)} \leq \|u\|_{H(t)}, \quad (5.1)$$

and

$$P_N^t u \rightarrow u \quad \text{in } H(t) \quad (5.2)$$

for all $u \in H(t)$. Lastly, we assume that

$$\|P_N^0 v\|_{V_0} \leq C \|v\|_{V_0} \quad \text{for all } v \in V_0. \quad (5.3)$$

Remark 5.9. We could have relaxed the definition of the spaces $V_j(t)$ and instead have asked for a family of finite-dimensional spaces $\{V_j(0)\}_{j \in \mathbb{N}}$ such that for all $j \in \mathbb{N}$,

- (i) $V_j(0) \subset V_0$
- (ii) $\dim(V_j) = j$
- (iii) $\bigcup_{i \in \mathbb{N}} V_i(0)$ is dense in V_0
- (iv) For every $v \in V_0$, there exists a sequence $\{v_j\}_{j \in \mathbb{N}}$ with $v_j \in V_j(0)$ such that $\|v_j - v\|_{V_0} \rightarrow 0$.

Furthermore, we can define the spaces $V_j(t) := \phi_t(V_j(0))$. The continuity of the map ϕ_t implies that these spaces share the same properties (with respect to $V(t)$) as the $V_j(0)$ given above; in particular the density result

$$\bigcup_{j \in \mathbb{N}} V_j(t) \quad \text{is dense in } V(t)$$

is true. Note that the basis of $V_j(t)$ does not necessarily have to be a subset of the basis of $V_{j+1}(t)$; this is the situation in finite element analysis, for example, so this relaxation can be useful for the purposes of numerical analysis. See [11, 12].

5.2 Galerkin approximation of (P)

Let $u_0 \in H_0$ and $f \in L^2_{V^*}$. The finite-dimensional approximation is to find a unique $u_N \in L^2_{V_N}$ with $\dot{u}_N \in L^2_{V_N}$ satisfying

$$\begin{aligned} l(t; \dot{u}_N(t), \chi_j^t) + a(t; u_N(t), \chi_j^t) + c(t; u_N(t), \chi_j^t) &= \langle f(t), \chi_j^t \rangle_{V^*(t), V(t)} \\ u_N(0) &= P_N^0(u_0) \end{aligned} \quad (5.4)$$

for all $j \in \{1, \dots, N\}$ and for almost every $t \in [0, T]$.

Theorem 5.10 (Existence and uniqueness of solutions to the finite-dimensional problem). There exists a unique $u_N \in L^2_{V_N}$ with $\dot{u}_N \in L^2_{V_N}$ satisfying the finite-dimensional problem (5.4). With $u_N(t) = \sum_{i=1}^N u_i^N(t) \chi_i^t$, the coefficient functions satisfy

$$\begin{aligned} u_i^N &\in C([0, T]) \\ \dot{u}_i^N &\in L^2(0, T). \end{aligned}$$

for all $i \in \{1, \dots, N\}$.

Proof. Substitute $u_N(t) = \sum_{i=1}^N u_i^N(t) \chi_i^t$ into (5.4) to yield

$$\sum_{i=1}^N \dot{u}_i^N(t) l_{ij}(t) + u_i^N(t) (a_{ij}(t) + c_{ij}(t)) = f_j(t) \quad (5.5)$$

with $l_{ij}(t) = l(t; \chi_i^t, \chi_j^t)$, $a_{ij}(t) = a(t; \chi_i^t, \chi_j^t)$, $c_{ij}(t) = c(t; \chi_i^t, \chi_j^t)$ and $f_j(t) = \langle f(t), \chi_j^t \rangle_{V^*(t), V(t)}$. Defining the vectors $(\mathbf{u}^N(\mathbf{t}))_i = u_i^N(t)$ and $(\mathbf{F}(\mathbf{t}))_i = f_i(t)$, and matrices $(\mathbf{L}(\mathbf{t}))_{ij} = l_{ji}(t)$, $(\mathbf{A}(\mathbf{t}))_{ij} = a_{ji}(t)$, $(\mathbf{C}(\mathbf{t}))_{ij} = c_{ji}(t)$, we can write (5.5) in matrix-vector form as

$$\mathbf{L}(\mathbf{t}) \dot{\mathbf{u}}^N(\mathbf{t}) + (\mathbf{A}(\mathbf{t}) + \mathbf{C}(\mathbf{t})) \mathbf{u}^N(\mathbf{t}) = \mathbf{F}(\mathbf{t}).$$

Elementary considerations show that $\mathbf{L}(\mathbf{t})$ is invertible and $\mathbf{L}(\cdot)^{-1} \in L^\infty(0, T; \mathbb{R}^{N \times N})$ so we can rearrange the system to

$$\dot{\mathbf{u}}^N(\mathbf{t}) + \mathbf{L}(\mathbf{t})^{-1} (\mathbf{A}(\mathbf{t}) + \mathbf{C}(\mathbf{t})) \mathbf{u}^N(\mathbf{t}) = \mathbf{L}(\mathbf{t})^{-1} \mathbf{F}(\mathbf{t}). \quad (5.6)$$

Note that $\mathbf{F}(\cdot) \in L^2(0, T; \mathbb{R}^N)$ and $\mathbf{A}(\cdot) + \mathbf{C}(\cdot) \in L^\infty(0, T; \mathbb{R}^{N \times N})$. So the coefficients of (5.6) are all measurable in time, and we can apply standard theory that guarantees the existence and uniqueness of $u_j^N \in C([0, T])$ (which are in fact absolutely continuous) with $\dot{u}_j^N \in L^2(0, T)$, and thus the existence and uniqueness of u_N . The function u_N is a solution in the sense that the derivative \dot{u}_N exists almost everywhere (and the ODE is satisfied almost everywhere), hence $u_N \in \tilde{C}_V^1$. \square

The Galerkin approximation is equivalent to the discrete equation

$$l(t; \dot{u}_N(t), v_N(t)) + a(t; u_N(t), v_N(t)) + c(t; u_N(t), v_N(t)) = (f(t), v_N(t))_{V^*(t), V(t)} \quad (\mathbf{P}_d)$$

for all $v_N \in L^2_{V_N}$. We look for *a priori* estimates on u_N and \dot{u}_N in appropriate norms.

Lemma 5.11 (A priori estimate on u_N). The following estimate holds:

$$\|u_N\|_{L_V^2} \leq C \left(\|u_0\|_{H_0} + \|f\|_{L_{V^*}^2} \right).$$

For convenience, we shall sometimes omit the argument (t) in expressions like $u_N(t)$. It should be clear from the context the instances in which we are referring to an element of $H(t)$ as opposed to an element of L_H^2 .

Proof. Picking $v_N = u_N$ in (5.7) gives

$$l(t; \dot{u}_N, u_N) + a(t; u_N, u_N) + c(t; u_N, u_N) = \langle f, u_N \rangle_{V^*(t), V(t)},$$

to which we apply the transport identity (L9) to yield

$$\frac{1}{2} \frac{d}{dt} l(t; u_N, u_N) + a(t; u_N, u_N) + c(t; u_N, u_N) - \frac{1}{2} m(t; u_N, u_N) = \langle f, u_N \rangle_{V^*(t), V(t)}.$$

Integrating in time and using the coercivity (L4) and boundedness (L3) of $l(t; \cdot, \cdot)$ leads to

$$\begin{aligned} \frac{C_c}{2} \|u_N(T)\|_{H(T)}^2 + \int_0^T a(t; u_N, u_N) + \int_0^T c(t; u_N, u_N) - \frac{1}{2} \int_0^T m(t; u_N, u_N) \\ \leq \int_0^T \langle f, u_N \rangle_{V^*(t), V(t)} + \frac{C_b}{2} \|u_N(0)\|_{H_0}^2, \end{aligned}$$

to which we use (A1) (the coercivity of $a(t; \cdot, \cdot)$), the boundedness of $c(t; \cdot, \cdot)$ and $m(t; \cdot, \cdot)$, and Young's inequality with $\epsilon > 0$:

$$\frac{C_c}{2} \|u_N(T)\|_{H(T)}^2 + \frac{C_1}{2} \|u_N\|_{L_V^2}^2 \leq \frac{C_2}{2} \|u_N\|_{L_H^2}^2 + \frac{1}{2\epsilon} \|f\|_{L_{V^*}^2}^2 + \frac{\epsilon}{2} \|u_N\|_{L_V^2}^2 + \frac{C_b}{2} \|u_N(0)\|_{H_0}^2.$$

That is,

$$C_c \|u_N(T)\|_{H(T)}^2 + (C_1 - \epsilon) \|u_N\|_{L_V^2}^2 \leq \frac{1}{\epsilon} \|f\|_{L_{V^*}^2}^2 + C_2 \|u_N\|_{L_H^2}^2 + C_b \|u_N(0)\|_{H_0}^2 \quad (5.7)$$

and if ϵ is small enough, we can rearrange and manipulate this to get

$$\|u_N(T)\|_{H(T)}^2 \leq C_3 \left(\|f\|_{L_{V^*}^2}^2 + \|u_N(0)\|_{H_0}^2 + \int_0^T \|u_N(t)\|_{H(t)}^2 \right).$$

Applying the integral form of Gronwall's inequality tells us that

$$\|u_N(t)\|_{H(t)}^2 \leq C_4 \left(\|f\|_{L_{V^*}^2}^2 + \|u_N(0)\|_{H_0}^2 \right).$$

Using this on (5.7), throwing away the first term on the left hand side and utilising (5.1), we get

$$\|u_N\|_{L_V^2} \leq C \left(\|u_0\|_{H_0} + \|f\|_{L_{V^*}^2} \right). \quad \square$$

Lemma 5.12 (A priori estimate on \dot{u}_N). If $u_0 \in V_0$ and $f \in L^2_H$ then the following estimate holds:

$$\|\dot{u}_N\|_{L^2_H} \leq C \left(\|u_0\|_{V_0} + \|f\|_{L^2_H} \right).$$

As before, we sometimes will not write the argument (t) in expressions like $u_N(t)$.

Proof. In (5.7), pick $v_N = \dot{u}_N$ and use (L4) to get

$$C_1 \|\dot{u}_N\|_{H(t)}^2 + a_s(t; u_N, \dot{u}_N) + a_n(t; u_N, \dot{u}_N) + c(t; u_N, \dot{u}_N) \leq (f, \dot{u}_N)_{H(t)}. \quad (5.8)$$

Then using assumption (A6), (5.8) is

$$C_1 \|\dot{u}_N\|_{H(t)}^2 + \frac{1}{2} \frac{d}{dt} a_s(t; u_N, u_N) \leq (f, \dot{u}_N)_{H(t)} + \frac{1}{2} r(t; u_N) - a_n(t; u_N, \dot{u}_N) - c(t; u_N, \dot{u}_N).$$

Integrating this yields

$$\begin{aligned} & C_1 \int_0^T \|\dot{u}_N\|_{H(t)}^2 + \frac{1}{2} a_s(T; u_N(T), u_N(T)) \\ & \leq \int_0^T (f, \dot{u}_N)_{H(t)} + \frac{1}{2} \int_0^T r(t; u_N) - \int_0^T a_n(t; u_N, \dot{u}_N) - \int_0^T c(t; u_N, \dot{u}_N) \\ & \quad + \frac{1}{2} a_s(0; u_N(0), u_N(0)). \end{aligned}$$

With (A5) (positivity of $a_s(t; \cdot, \cdot)$), the bound (A4) on $a_s(0; \cdot, \cdot)$, the bound (A7) on $r(t; \cdot)$, the bound (A3) on $a_n(t; \cdot, \cdot)$, the bound on $c(t; \cdot, \cdot)$ and Young's inequality with $\epsilon > 0$ and $\delta > 0$, we get

$$\begin{aligned} C_1 \|\dot{u}_N\|_{L^2_H}^2 & \leq \frac{1}{2\delta} \|f\|_{L^2_H}^2 + \left(C_2 + \frac{C_3}{2\epsilon} \right) \|u_N\|_{L^2_V}^2 + \frac{(\delta + C_3\epsilon)}{2} \|\dot{u}_N\|_{L^2_H}^2 + C_4 \|u_N(0)\|_{V_0}^2 \\ & \leq \frac{1}{2\delta} \|f\|_{L^2_H}^2 + C_5 \left(C_2 + \frac{C_3}{2\epsilon} \right) (\|u_N(0)\|_{H_0}^2 + \|f\|_{L^2_H}^2) + \frac{(\delta + C_3\epsilon)}{2} \|\dot{u}_N\|_{L^2_H}^2 \\ & \quad + C_4 \|u_N(0)\|_{V_0}^2 \quad \text{(by the first a priori bound)} \\ & = \left(\frac{1}{2\delta} + C_5 \left(C_2 + \frac{C_3}{2\epsilon} \right) \right) \|f\|_{L^2_H}^2 + C_5 \left(C_2 + \frac{C_3}{2\epsilon} \right) \|u_N(0)\|_{H_0}^2 \\ & \quad + \frac{(\delta + C_3\epsilon)}{2} \|\dot{u}_N\|_{L^2_H}^2 + C_4 \|u_N(0)\|_{V_0}^2. \end{aligned}$$

So if ϵ and δ are small enough,

$$\|\dot{u}_N\|_{L^2_H}^2 \leq C_7 \left(\|f\|_{L^2_H}^2 + \|u_0\|_{V_0}^2 \right),$$

where we used the bound (5.3) on $u_N(0)$. □

5.3 Second proof of existence for (P)

Sketch proof of Theorem 3.6. The uniform bound

$$\|u_N\|_{L_V^2} \leq C \left(\|u_0\|_{H_0} + \|f\|_{L_{V^*}^2} \right)$$

implies that

$$u_N \rightharpoonup u \quad \text{in } L_V^2 \quad (5.9)$$

for some $u \in L_V^2$.

Picking in (P_d) $v_N = \chi_j^t$, where $j \in \{0, \dots, N\}$, and multiplying by $\zeta \in C^1[0, T]$ with $\zeta(T) = 0$, we get

$$l(t; \dot{u}_N(t), \zeta(t)\chi_j^t) + a(t; u_N(t), \zeta(t)\chi_j^t) + c(t; u_N(t), \zeta(t)\chi_j^t) = \langle f(t), \zeta(t)\chi_j^t \rangle_{V^*(t), V(t)},$$

and then using the transport formula (L9), integrating, and passing to the limit with the help of (5.9) and (5.2):

$$\begin{aligned} & - \int_0^T l(t; u(t), \zeta'(t)\chi_j^t) + \int_0^T a(t; u(t), \zeta(t)\chi_j^t) + \int_0^T c(t; u(t), \zeta(t)\chi_j^t) - \int_0^T m(t; u(t), \zeta(t)\chi_j^t) \\ & = \int_0^T \langle f(t), \zeta(t)\chi_j^t \rangle_{V^*(t), V(t)} + l(0; u_0, \zeta(0)\chi_j^0). \end{aligned} \quad (5.10)$$

Now, because $\{\chi_j^0\}_{j \in \mathbb{N}}$ is a basis for V_0 , we can write an arbitrary element of V_0 as $v = \sum_{i=1}^{\infty} \alpha_j \chi_j^0$. By definition, the sequence $v_n = \sum_{i=1}^n \alpha_j \chi_j^0$ converges to v in V_0 . It follows that

$$\phi_t v_n = \sum_{j=1}^n \alpha_j \chi_j^t \rightarrow \phi_t v \quad \text{in } V(t).$$

Letting $\zeta(0) = 0$ and multiplying (5.10) by α_j and summing over j gives us

$$\int_0^T \zeta'(t) l(t; u(t), \phi_t v_n) = - \int_0^T \zeta(t) \langle f(t) - \mathcal{A}(t)u(t) - \mathcal{C}(t)u(t) + \mathcal{M}(t)u(t), \phi_t v_n \rangle_{V^*(t), V(t)}. \quad (5.11)$$

It is not difficult to see that the dominated convergence theorem applies and we can pass to the limit in (5.11) to obtain

$$\int_0^T \zeta'(t) l(t; u(t), \phi_t v) = - \int_0^T \zeta(t) \langle f(t) - \mathcal{A}(t)u(t) - \mathcal{C}(t)u(t) + \mathcal{M}(t)u(t), \phi_t v \rangle_{V^*(t), V(t)}.$$

If we further let $\zeta \in \mathcal{D}(0, T)$, this is precisely the statement

$$\frac{d}{dt} l(t; u(t), \phi_t v) = \langle f(t) - \mathcal{A}(t)u(t) - \mathcal{C}(t)u(t) + \mathcal{M}(t)u(t), \phi_t v \rangle_{V^*(t), V(t)}$$

in the weak sense. This is true for every $v \in V_0$, and because $f - \mathcal{A}u - \mathcal{C}u \in L^2_{V^*}$, by Lemma 3.5,

$$\mathcal{L}\dot{u} + \mathcal{A}u + \mathcal{C}u = f$$

holds as an equality in $L^2_{V^*}$ with $u \in W(V, V^*)$.

Let us now check the initial condition. Let $w \in V_0$, take $\zeta \in C^1[0, T]$ with $\zeta(T) = 0$, and set $v(t) = \zeta(t)\phi_t w$. We see that $v \in L^2_V$. Since $w \in V_0$, there exist coefficients α_j with $w = \sum_{j=1}^{\infty} \alpha_j \chi_j^0$. So

$$v(t) = \zeta(t) \sum_{j=1}^{\infty} \alpha_j \chi_j^t. \quad (5.12)$$

The sequence $\{v_N\}_{N \in \mathbb{N}}$ defined by

$$v_N(t) = \zeta(t) \sum_{j=1}^N \alpha_j \chi_j^t \quad (5.13)$$

is such that $v_N \in L^2_{V_N}$ and satisfies $\|v_N - v\|_{L^2_V} \rightarrow 0$ because

$$\|v_N - v\|_{L^2_V}^2 = \int_0^T \|\zeta(t) \left(\sum_{j=1}^N \alpha_j \chi_j^t - \phi_t w \right)\|_{V(t)}^2 dt \leq C \left\| \sum_{j=1}^N \alpha_j \chi_j^0 - w \right\|_{V_0}^2$$

which converges to zero by definition of w as an infinite sum. Similarly, we can show that $\dot{v}_N \rightarrow \dot{v}$ in L^2_V .

Using the identity (L9) with v chosen as in (5.12), we see that **(P')** is alternatively

$$\begin{aligned} \frac{d}{dt} l(t; u(t), v(t)) + a(t; u(t), v(t)) + c(t; u(t), v(t)) \\ = \langle f(t), v(t) \rangle_{V^*(t), V(t)} + l(t; \dot{v}(t), u(t)) + m(t; u(t), v(t)), \end{aligned}$$

which we integrate to get

$$\begin{aligned} \int_0^T a(t; u(t), v(t)) + c(t; u(t), v(t)) dt \\ = \int_0^T \langle f(t), v(t) \rangle_{V^*(t), V(t)} + l(t; u(t), \dot{v}(t)) + m(t; u(t), v(t)) dt + l(0; u(0), v(0)). \end{aligned} \quad (5.14)$$

Similarly, with v_N chosen as in (5.13) in the Galerkin equation (5.7), to which we again apply (L9) and integrate to obtain

$$\begin{aligned} \int_0^T a(t; u_N(t), v_N(t)) + c(t; u_N(t), v_N(t)) dt \\ = \int_0^T \langle f(t), v_N(t) \rangle_{V^*(t), V(t)} + l(t; u_N(t), \dot{v}_N(t)) + m(t; u_N(t), v_N(t)) dt + l(0; u_N(0), v_N(0)). \end{aligned}$$

Using $u_N \rightharpoonup u$, $v_N \rightarrow v$, and $\dot{v}_N \rightarrow \dot{v}$, we may pass to the limit in this equation and a comparison of the result to (5.14) will tell us that

$$l(0; u_0 - u(0), \zeta(0)w) = 0.$$

The arbitrariness of $w \in V_0$, and the density of V_0 in H_0 yield the result.

That the solution is unique follows by a straightforward adaptation of the standard technique for linear parabolic PDEs. \square

5.4 Proof of regularity

If $u_0 \in V_0$ and $f \in L^2_H$ then we may use the estimates of Lemma 5.12 and obtain the convergence

$$\begin{aligned} u_N &\rightharpoonup u && \text{in } L^2_V \\ \dot{u}_N &\rightharpoonup w && \text{in } L^2_H \end{aligned} \quad (5.15)$$

for some $u \in L^2_V$ and $w \in L^2_{V^*}$ and for a subsequence which we have relabelled. Now we show that in fact, $w = \dot{u}$.

Lemma 5.13. In the context of the above convergence results, $w = \dot{u}$.

Proof. By definition

$$\int_0^T \langle \dot{u}_N(t), \eta(t) \rangle_{V^*(t), V(t)} = - \int_0^T (u_N(t), \dot{\eta}(t))_{H(t)} - \int_0^T c(t; u_N(t), \eta(t)) \quad (5.16)$$

holds for all $\eta \in \mathcal{D}_V(0, T)$. Note that

$$\langle \cdot, \eta \rangle_{L^2_{V^*}, L^2_V}, \quad (\cdot, \dot{\eta})_{L^2_H}, \quad \langle \mathcal{C}(\cdot), \eta \rangle_{L^2_{V^*}, L^2_V}$$

are all elements of $L^2_{V^*}$. Using (5.15), we can then pass to the limit in (5.16) to obtain

$$\int_0^T \langle w(t), \eta(t) \rangle_{V^*(t), V(t)} = - \int_0^T (u(t), \dot{\eta}(t))_{H(t)} - \int_0^T c(t; u(t), \eta(t)),$$

i.e., $w = \dot{u}$. \square

Proof of Theorem 3.10. So we have the convergence

$$\begin{aligned} u_N &\rightharpoonup u && \text{in } L^2_V \\ \dot{u}_N &\rightharpoonup \dot{u} && \text{in } L^2_H. \end{aligned}$$

Given $v \in L^2_V$, by density, there is a sequence $\{v_M\}_M$ with $v_M \in L^2_{V_M}$ for each M such that

$$v_M(t) = \sum_{j=1}^M \alpha_j^M(t) \chi_j^t \quad \text{and} \quad \|v_M - v\|_{L^2_V} \rightarrow 0.$$

For $j = 1, \dots, N$, consider the equation (5.4):

$$l(t; \dot{u}_N(t), \chi_j^t) + a(t; u_N(t), \chi_j^t) + c(t; u_N(t), \chi_j^t) = (f(t), \chi_j^t)_{H(t)}.$$

If $M \leq N$, then $v_M \in L_{V_N}^2$ and we multiply the above by $\alpha_j^M(t)$ and sum up to get

$$l(t; \dot{u}_N(t), v_M(t)) + a(t; u_N(t), v_M(t)) + c(t; u_N(t), v_M(t)) = (f(t), v_M(t))_{H(t)}.$$

Multiplying this equation by $\zeta \in \mathcal{D}(0, T)$ and integrating:

$$\begin{aligned} \int_0^T \zeta(t) (l(t; \dot{u}_N(t), v_M(t)) + a(t; u_N(t), v_M(t)) + c(t; u_N(t), v_M(t))) \\ = \int_0^T \zeta(t) (f(t), v_M(t))_{H(t)}. \end{aligned}$$

By the bounds on the respective bilinear forms, we see that

$$\begin{aligned} \langle \mathcal{L}(\cdot), \zeta v_M \rangle_{L_{V^*}^2, L_V^2} &\in L_{H^*}^2 \\ \langle \mathcal{A}(\cdot), \zeta v_M \rangle_{L_{V^*}^2, L_V^2} &\in L_{V^*}^2 \\ \langle \mathcal{C}(\cdot), \zeta v_M \rangle_{L_{V^*}^2, L_V^2} &\in L_{H^*}^2, \end{aligned}$$

so we obtain in the limit as $N \rightarrow \infty$ the equation

$$\int_0^T \zeta(t) (l(t; \dot{u}(t), v_M(t)) + a(t; u(t), v_M(t)) + c(t; u(t), v_M(t))) = \int_0^T \zeta(t) (f(t), v_M(t))_{H(t)}.$$

Now note that as a function of v_M , each term in the above equation is an element of $L_{V^*}^2$ again because of the bounds on $l(t; \cdot, \cdot)$, $a(t; \cdot, \cdot)$ and $c(t; \cdot, \cdot)$. So we send $M \rightarrow \infty$, bearing in mind that v_M strongly converges to v in L_V^2 :

$$\int_0^T \zeta(t) (l(t; \dot{u}(t), v(t)) + a(t; u(t), v(t)) + c(t; u(t), v(t))) = \int_0^T \zeta(t) (f(t), v(t))_{H(t)}.$$

Therefore, we have

$$l(t; \dot{u}(t), v(t)) + a(t; u(t), v(t)) + c(t; u(t), v(t)) = (f(t), v(t))_{H(t)}$$

for every $v \in L_V^2$ for almost every $t \in [0, T]$. Hence $u \in W(V, H)$ is a solution. The stability estimate follows directly from the estimates in Lemmas 5.11 and 5.12. \square

6 Applications to evolving hypersurfaces

Our applications rely on Sobolev spaces defined on hypersurfaces. For reasons of space we shall only briefly discuss the theory here and refer the reader to [13, 9, 35, 19, 32] for more details on

analysis on surfaces. We emphasise the text [32] which contains a detailed overview of the essential facts of Sobolev spaces on hypersurfaces.

Suppose that Γ is an n -dimensional C^k hypersurface in \mathbb{R}^{n+1} with $k \geq 2$ and smooth boundary $\partial\Gamma$. Throughout we assume that Γ is orientable with unit normal ν . We can define $L^2(\Gamma)$ in the natural way: it consists of the set of measurable functions $f: \Gamma \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^2(\Gamma)} := \left(\int_{\Gamma} |f(x)|^2 d\sigma(x) \right)^{\frac{1}{2}} < \infty,$$

where $d\sigma$ is the surface measure on Γ (which we often omit writing). We will use the notation $\nabla_{\Gamma} = (\underline{D}_1, \dots, \underline{D}_{n+1})$ to stand for the surface gradient on a hypersurface Γ , and $\Delta_{\Gamma} := \nabla_{\Gamma} \cdot \nabla_{\Gamma}$ will denote the Laplace–Beltrami operator. The integration by parts formula for functions $f \in C^1(\bar{\Gamma}; \mathbb{R}^{n+1})$ is

$$\int_{\Gamma} \nabla_{\Gamma} \cdot f = \int_{\Gamma} f \cdot H\nu + \int_{\partial\Gamma} f \cdot \mu$$

where μ is the unit conormal vector which is normal to $\partial\Gamma$ and tangential to Γ . Now if $\psi \in C_c^1(\Gamma)$, then this formula implies

$$\int_{\Gamma} f \underline{D}_i \psi = - \int_{\Gamma} \psi \underline{D}_i f + \int_{\Gamma} f \psi H\nu_i,$$

with the boundary term disappearing due to the compact support. This relation is the basis for defining weak derivatives. We say $f \in L^2(\Gamma)$ has weak derivative $g_i =: \underline{D}_i f \in L^2(\Gamma)$ if for every $\psi \in C_c^1(\Gamma)$,

$$\int_{\Gamma} f \underline{D}_i \psi = - \int_{\Gamma} \psi g_i + \int_{\Gamma} f \psi H\nu_i$$

holds. Then we can define the Sobolev space

$$H^1(\Gamma) = \{f \in L^2(\Gamma) \mid \underline{D}_i f \in L^2(\Gamma), i = 1, \dots, n+1\}$$

with the norm

$$\|f\|_{H^1(\Gamma)} = \sqrt{\|f\|_{L^2(\Gamma)}^2 + \|\nabla_{\Gamma} f\|_{L^2(\Gamma)}^2}.$$

We shall also need a fractional-order Sobolev space. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Define the space

$$H^{\frac{1}{2}}(\partial\Omega) = \{u \in L^2(\partial\Omega) \mid \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^n} d\sigma(x) d\sigma(y) < \infty\}.$$

This is a Hilbert space with the inner product

$$\begin{aligned} (u, v)_{H^{\frac{1}{2}}(\partial\Omega)} &= \int_{\partial\Omega} u(x)v(x) d\sigma(x) \\ &+ \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^n} d\sigma(x) d\sigma(y). \end{aligned}$$

See [32, §2.4] and [10, §3.2] for details. The notation

$$|u|_{H^{\frac{1}{2}}(\Gamma_0)} = \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^n} d\sigma(x) d\sigma(y) \right)^{\frac{1}{2}}$$

for the seminorm is convenient.

Now, recall Green's formula:

$$- \int_{\Omega} \Delta v w = \int_{\Omega} \nabla v \nabla w - \int_{\partial\Omega} \nabla v \cdot \nu w.$$

This allows us to define a normal derivative of functions in the space

$$\{v \in H^1(\Omega) \mid \Delta v \in H^{-1}(\Omega)\}$$

as the element $\frac{\partial v}{\partial \nu} \in H^{-\frac{1}{2}}(\partial\Omega)$ determined by

$$\left\langle \frac{\partial v}{\partial \nu}, w \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} := \int_{\Omega} \nabla v \nabla \mathbb{E}(w) + \langle \Delta v, \mathbb{E}(w) \rangle_{H^{-1}(\Omega), H^1(\Omega)} \quad (6.1)$$

where $\mathbb{E}(w) \in H^1(\Omega)$ is an extension of $w \in H^{\frac{1}{2}}(\partial\Omega)$. The functional $\frac{\partial v}{\partial \nu}$ is independent of the extension used for w .

6.1 Evolving spatial domains

We want to showcase the following four examples that demonstrate the applicability of our theory in different situations:

1. A surface heat equation on an evolving compact hypersurface without boundary,

and the following on an evolving flat hypersurface with boundary:

2. A bulk equation
3. A coupled bulk-surface system
4. A problem with dynamic boundary conditions

We first discuss evolving compact hypersurfaces and evolving flat hypersurfaces with boundary in the context of §2.

6.1.1 Evolving compact hypersurfaces

For each $t \in [0, T]$, let $\Gamma(t) \subset \mathbb{R}^{n+1}$ be a compact (i.e., no boundary) n -dimensional hypersurface of class C^2 , and assume the existence of a flow $\Phi: [0, T] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that for all $t \in [0, T]$, with $\Gamma_0 := \Gamma(0)$, the map $\Phi_t^0(\cdot) := \Phi(t, \cdot): \Gamma_0 \rightarrow \Gamma(t)$ is a C^2 -diffeomorphism that satisfies

$$\begin{aligned} \frac{d}{dt} \Phi_t^0(\cdot) &= \mathbf{w}(t, \Phi_t^0(\cdot)) \\ \Phi_0^0(\cdot) &= \text{Id}(\cdot). \end{aligned} \tag{6.2}$$

We think of the map $\mathbf{w}: [0, T] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ as a velocity field, and we assume that it is C^2 and satisfies the uniform bound

$$|\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)| \leq C \quad \text{for all } t \in [0, T].$$

A normal vector field on the hypersurfaces is denoted by $\nu: [0, T] \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$.

Let $V(t) = H^1(\Gamma(t))$ and $H(t) = L^2(\Gamma(t))$. It is known that the embeddings $V(t) \subset H(t) \subset V^*(t)$ form a separable Hilbert triple. We define the pullback operator by

$$\phi_{-t}v = v \circ \Phi_t^0.$$

By [34, Lemma 3.2], the map ϕ_{-t} is such that

$$\phi_{-t}: L^2(\Gamma(t)) \rightarrow L^2(\Gamma_0) \quad \text{and} \quad \phi_{-t}: H^1(\Gamma(t)) \rightarrow H^1(\Gamma_0)$$

are linear homeomorphisms with the constants of continuity not dependent on t . We denote by $\phi_{-t}^*: H^{-1}(\Gamma_0) \rightarrow H^{-1}(\Gamma(t))$ the dual operator. The maps $t \mapsto \|\phi_t u\|_{X(t)}$ (for $X = L^2$ and H^1) are continuous [34, Lemma 3.3], thus we have compatibility of the pairs $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$, and the spaces $L_{L^2}^2$, $L_{H^1}^2$ and $L_{H^{-1}}^2$ are well-defined.

Let us now work out a formula for the strong material derivative. Note that, by the smoothness of $\Gamma(t)$, any function $u: \Gamma(t) \rightarrow \mathbb{R}$ can be extended to a neighbourhood of the space time surface $\cup_{t \in [0, T]} \Gamma(t) \times \{t\}$ in \mathbb{R}^{n+2} in which ∇u and u_t for the extension are well-defined. Suppose that $u \in C_V^1$. The derivative of its pullback is

$$\begin{aligned} \frac{d}{dt} \phi_{-t}u(t) &= \frac{d}{dt} u(t, \Phi_t^0(y)) = u_t(t, \Phi_t^0(y)) + \nabla u|_{(t, \Phi_t^0(y))} \cdot \mathbf{w}(t, \Phi_t^0(y)) \\ &= \phi_{-t}u_t(t, y) + \phi_{-t}(\nabla u(t, y)) \cdot \phi_{-t}(\mathbf{w}(t, y)), \quad y \in \Gamma_0 \end{aligned}$$

giving

$$\dot{u}(t, x) = u_t(t, x) + \nabla u(t, x) \cdot \mathbf{w}(t, x), \quad x \in \Gamma(t). \tag{6.3}$$

The expression on the right hand side is independent of the extension.

It is clear that our definition of the strong material derivative coincides with the well-established definition (see §2.4). Moreover, as we mentioned in the introduction, we can see from (6.3) that the

material derivative depends on the evolution of the hypersurface, which in turn depends on the map Φ_t^0 .

We denote by J_t^0 the change of area element when transforming from Γ_0 to $\Gamma(t)$, i.e., for any integrable function $\zeta: \Gamma(t) \rightarrow \mathbb{R}$

$$\int_{\Gamma(t)} \zeta = \int_{\Gamma_0} (\zeta \circ \Phi_t^0) J_t^0 = \int_{\Gamma_0} \phi_{-t} \zeta J_t^0.$$

Using the transport identity

$$\frac{d}{dt} \int_{G(t)} \zeta(t) \Big|_t = \int_{G(t)} \dot{\zeta}(t) + \zeta(t) \nabla_{G(t)} \cdot \mathbf{w}(t)$$

on any portion $G \subset \Gamma$ with points that move with the velocity field \mathbf{w} (for instance, see ([11]) one can easily show that

$$\frac{d}{dt} J_t^0 = \phi_{-t} (\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0. \quad (6.4)$$

The field J_t^0 is uniformly bounded by positive constants

$$\frac{1}{C_J} \leq J_t^0(z) \leq C_J \quad \text{for all } z \in \Gamma_0 \text{ and for all } t \in [0, T].$$

The $L^2(\Gamma(t))$ inner product is

$$\begin{aligned} (u, v)_{L^2(\Gamma(t))} &= \int_{\Gamma(t)} u(x)v(x) = \int_{\Gamma_0} (u \circ \Phi_t^0(z))(v \circ \Phi_t^0(z)) J_t^0(z) \\ &= \int_{\Gamma_0} \phi_{-t} u \phi_{-t} v J_t^0, \end{aligned}$$

where we made the substitution $x = \Phi_t^0(z)$. The bilinear form $\hat{b}(t; \cdot, \cdot): H_0 \times H_0 \rightarrow \mathbb{R}$ (which satisfies $(u, v)_{H(t)} = \hat{b}(\phi_{-t} u, \phi_{-t} v)$ by definition) is

$$\hat{b}(t; u_0, v_0) = \int_{\Gamma_0} u_0 v_0 J_t^0,$$

so the action of the operator $T_t: H_0 \rightarrow H_0$ (see Definition 2.22 and Theorem 2.32) is just pointwise multiplication:

$$T_t u_0 = J_t^0 u_0.$$

With this, we see that the function θ from Assumptions 2.24 is

$$\begin{aligned} \theta(t, u_0) &:= \frac{d}{dt} \|\phi_t u_0\|_{L^2(\Gamma(t))}^2 = \frac{d}{dt} \int_{\Gamma_0} u_0^2 J_t^0 = \int_{\Gamma_0} u_0^2 \phi_{-t} (\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0 \\ &= \int_{\Gamma(t)} (\phi_t u_0)^2 \nabla_{\Gamma} \cdot \mathbf{w}(t) \end{aligned}$$

where the cancellation of the Jacobian terms in the last equality is thanks to the inverse function theorem. Now, $v \mapsto \theta(t, v)$ is continuous because if $v_n \rightarrow v$ in $L^2(\Gamma_0)$, then $v_n^2 \rightarrow v^2$ in $L^1(\Gamma_0)$ and so

$$\begin{aligned} |\theta(t, v_n) - \theta(t, v)| &\leq \int_{\Gamma_0} |v_n^2 - v^2| |\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0| \\ &\leq C \|v_n^2 - v^2\|_{L^1(\Gamma_0)} \\ &\rightarrow 0. \end{aligned}$$

Finally,

$$\begin{aligned} |\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| &= 4 \int_{\Gamma(t)} \phi_t u_0 \phi_t v_0 \nabla_{\Gamma(t)} \cdot \mathbf{w}(t) \\ &\leq C \|u_0\|_{L^2(\Gamma_0)} \|v_0\|_{L^2(\Gamma_0)}. \end{aligned}$$

So we have checked Assumptions 2.24. Now if $u_0, v_0 \in L^2(\Gamma_0)$,

$$\hat{c}(t; u_0, v_0) = \frac{\partial}{\partial t} \hat{b}(t; u_0, v_0) = \int_{\Gamma_0} u_0 v_0 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}) J_t^0,$$

thus the bilinear form $c(t; \cdot, \cdot)$ of Definition 2.25 is

$$c(t; u, v) = \int_{\Gamma_0} \phi_{-t} u \phi_{-t} v \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}) J_t^0 = \int_{\Gamma(t)} uv \nabla_{\Gamma(t)} \cdot \mathbf{w},$$

which, as claimed in Lemma 2.26, is measurable in t and bounded on $H(t) \times H(t)$. So then $u \in L_V^2$ has a weak material derivative $\dot{u} \in L_{V^*}^2$ if and only if

$$\int_0^T \langle \dot{u}(t), \eta(t) \rangle_{V^*(t), V(t)} = - \int_0^T \int_{\Gamma(t)} u(t) \dot{\eta}(t) - \int_0^T \int_{\Gamma(t)} u(t) \eta(t) \nabla_{\Gamma(t)} \cdot \mathbf{w}(t)$$

holds for all $\eta \in \mathcal{D}_V(0, T)$ (cf. [34, 27]).

Finally, [34, Lemma 3.6] proves that $T_{(\cdot)} u(\cdot) \in \mathcal{W}(V_0, V_0^*)$ if and only if $u(\cdot) \in \mathcal{W}(V_0, V_0^*)$, due to the fact that both $J_{(\cdot)}^0$ and its reciprocal $1/J_{(\cdot)}^0$ are in $C^1([0, T] \times \Gamma_0)$. To see that the evolving space equivalence (Assumption 2.31) holds, take $u \in \mathcal{W}(V_0, V_0^*)$ and obtain by the product rule and (6.4) the identity

$$(J_t^0 u(t))' = J_t^0 u'(t) + \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}) J_t^0 u(t).$$

Therefore, the maps $\hat{S}(t)$ and $\hat{D}(t)$ (from Theorem 2.32) are $\hat{S}(t)u'(t) = J_t^0 u'(t)$ and $\hat{D}(t) \equiv 0$. It follows by the uniform bound on J_t^0 that $\hat{S}(\cdot)u'(\cdot) \in L^2(0, T; V_0^*)$. By Theorem 2.32, we have that the space $W(V, V^*) = \{u \in L_{H^1}^2 \mid \dot{u} \in L_{H^{-1}}^2\}$ is indeed isomorphic to $\mathcal{W}(V_0, V_0^*)$ and there is an equivalence of norms between

$$\|u\|_{W(V, V^*)} \quad \text{and} \quad \|\phi_{-(\cdot)} u(\cdot)\|_{\mathcal{W}(V_0, V_0^*)}.$$

It is easy to see that $W(V, H)$ and $\mathcal{W}(V_0, H_0)$ are also equivalent.

Remark 6.1 (Velocity fields). It is useful to note that there are different notions of velocities for an evolving hypersurface.

- A hypersurface defined by a level set function, $\Gamma(t) := \{x \mid \Psi(x, t) = 0\}$, has a *normal velocity*

$$\mathbf{w}_\nu = -\frac{\Psi_t}{|\nabla\Psi|} \cdot \frac{\nabla\Psi}{|\nabla\Psi|}.$$

The normal velocity is sufficient to define the evolution of the hypersurface from $\Gamma(0)$.

- In applications there may be a *physical velocity*

$$\mathbf{w}_\nu + \mathbf{w}_\tau$$

where \mathbf{w}_ν is the normal component and \mathbf{w}_τ is the tangential component. The tangential velocity may be associated with the motion of material points and may be relevant to the mathematical models of processes on the surface; for example yielding an advective flux.

- The velocity field (6.2) defined earlier defines the path of points on the initial surface. This velocity arising from parametrising the surface

$$\mathbf{w}_\nu + \mathbf{w}_a$$

has \mathbf{w}_a as an *arbitrary tangential velocity* that may or may not coincide with the physical tangential velocity \mathbf{w}_τ . In finite element analysis, it may be necessary to choose the tangential velocity \mathbf{w}_a in an ALE approach so as to yield a shape-regular or adequately refined mesh. See [15] and [13, §5.7] for more details on this.

- Now suppose that \mathbf{w} is purely tangential (so $\mathbf{w} \cdot \nu = 0$). In this case, material points on the initial surface get transported across the surface as time increases but *the surface remains the same*. One can see this for a sufficiently smooth initial surface Γ_0 by supposing that Γ_0 is the zero-level set of a function $\Psi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$:

$$\Gamma_0 = \{x \in \mathbb{R}^{n+1} \mid \Psi(x) = 0\}.$$

Let P be a material point on Γ_0 and $\gamma(t)$ denote the position of P at time t , with $\gamma(t) \in \Gamma(t)$. Then a purely tangential velocity means that $\nabla\Psi(\gamma(t)) \cdot \gamma'(t) = 0$, but this is precisely

$$\frac{d}{dt}\Psi(\gamma(t)) = 0,$$

so the point persists in being a zero of the level set. Since P was arbitrary, we conclude that $\Gamma(t)$ coincides with Γ_0 for all $t \in [0, T]$, i.e.,

$$\Gamma(t) = \{x \in \mathbb{R}^{n+1} \mid \Psi(x) = 0\}.$$

- In certain situations, it can be useful to consider on an evolving surface a *boundary velocity* \mathbf{w}_b which we can extend (arbitrarily) to the interior. In the case of flat hypersurfaces, $\mathbf{w}_\nu \equiv 0$ since there is movement in the conormal direction and no movement in the normal direction. The conormal component of the arbitrary velocity must agree with the conormal component of the boundary velocity \mathbf{w}_b , otherwise the velocities map to two different surfaces.

Remark 6.2 (Normal time derivative). Suppose that the velocity field associated to the evolving hypersurface $\{\Gamma(t)\}$ is $\mathbf{w} = \mathbf{w}_\nu + \mathbf{w}_\tau$ where \mathbf{w}_ν is a normal velocity field and \mathbf{w}_τ is a tangential velocity field. In this case, the formula

$$\partial^\circ u = u_t + \nabla u \cdot \mathbf{w}_\nu$$

defines the *normal time derivative* $\partial^\circ u$.

6.1.2 Evolving flat hypersurfaces with boundary

Three of our applications are on evolving domains in \mathbb{R}^n . We discuss here what is common to the examples and leave the specifics and peculiarities to be detailed on a case-by-case basis as required.

For each $t \in [0, T]$, let $\Omega(t) \subset \mathbb{R}^n$ be a bounded open and connected domain of class C^2 with boundary $\{\Gamma(t)\}_{t \in [0, T]}$. We may view $\Omega(t)$ as an evolving flat hypersurface in \mathbb{R}^{n+1} and $\Gamma(t)$ as an evolving compact $(n-1)$ -dimensional hypersurface in \mathbb{R}^n . We denote $\Omega_0 := \Omega(0)$ and $\Gamma_0 := \Gamma(0)$. For each $t \in [0, T]$, we assume the existence of a map $\Phi_t^0: \overline{\Omega_0} \rightarrow \overline{\Omega(t)}$ such that $\Phi_t^0(\Omega_0) = \Omega(t)$ and $\Phi_t^0(\Gamma_0) = \Gamma(t)$ (i.e., interiors are mapped to interiors and boundaries are mapped to boundaries). We assume that

$$\begin{aligned} \Phi_t^0: \Omega_0 &\rightarrow \Omega(t) \text{ is a } C^2\text{-diffeomorphism,} \\ \Phi_t^0: \Gamma_0 &\rightarrow \Gamma(t) \text{ is a } C^2\text{-diffeomorphism,} \\ \Phi_{(\cdot)}^0 &\in C^2([0, T] \times \Omega_0), \end{aligned}$$

and that $\Omega(t)$ and $\Gamma(t)$ evolve with velocity \mathbf{w} as in (6.2).

Definition 6.3. For functions $u: \Omega_0 \rightarrow \mathbb{R}$ and $v: \Gamma_0 \rightarrow \mathbb{R}$, define the maps

$$\begin{aligned} \phi_{\Omega, t} u &= u \circ \Phi_0^t|_{\Omega_0} \\ \phi_{\Gamma, t} v &= v \circ \Phi_0^t|_{\Gamma_0}. \end{aligned}$$

We find that

$$\phi_{\Omega, t}: H^1(\Omega_0) \rightarrow H^1(\Omega(t)) \quad \text{and} \quad \phi_{\Omega, t}: L^2(\Omega_0) \rightarrow L^2(\Omega(t))$$

are linear homeomorphisms with the constants of continuity not depending on t (we can either adapt the proofs in [34] or use Problem 1.3.1 in [26]). Furthermore, since the boundary $\Gamma(t)$ is a C^2 hypersurface, it satisfies the assumptions in §6.1.1 and so it follows that the maps

$$\phi_{\Gamma, t}: H^1(\Gamma_0) \rightarrow H^1(\Gamma(t)) \quad \text{and} \quad \phi_{\Gamma, t}: L^2(\Gamma_0) \rightarrow L^2(\Gamma(t))$$

are also linear homeomorphisms with the constants of continuity not depending on t .

One of the most important terms in the solution space regime is the Jacobian $J_{\Omega,(\cdot)}^0 := \det \mathbf{D}\Phi_{(\cdot)}^0 \in C^1([0, T] \times \Omega_0)$; one can show that it satisfies much of the same properties (see [5] for this) as the Jacobian term did in §6.1.1 for the case of compact hypersurfaces. Hence it is straightforward to adapt the proofs for the case of a domain with boundary to yield the fulfilment of the evolving space equivalence Assumption 2.35 between $\mathcal{W}(H^1(\Omega_0), L^2(\Omega_0))$ and $W(H_\Omega^1, L_\Omega^2)$.

6.2 Application 1: the surface advection-diffusion equation

Suppose that we have an evolving hypersurface $\Gamma(t)$ that evolves with normal velocity \mathbf{w}_ν . Given a surface flux \mathbf{q} , we consider the conservation law

$$\frac{d}{dt} \int_{M(t)} u = - \int_{\partial M(t)} \mathbf{q} \cdot \mu$$

on an arbitrary portion $M(t) \subset \Gamma(t)$, where μ denotes the conormal on $\partial M(t)$. Without loss of generality we can assume that \mathbf{q} is tangential. This conservation law implies the pointwise equation

$$u_t + \nabla u \cdot \mathbf{w}_\nu + u \nabla_\Gamma \cdot \mathbf{w}_\nu + \nabla_\Gamma \cdot \mathbf{q} = 0.$$

Assuming that the flux is a combination of a diffusive flux and an advective flux

$$\mathbf{q} = -\nabla_\Gamma u + u \mathbf{b}_\tau$$

where \mathbf{b}_τ is an advective tangential velocity field, we obtain

$$\dot{u} - \Delta_\Gamma u + u \nabla_\Gamma \cdot \mathbf{w} + \nabla_\Gamma u \cdot (\mathbf{b}_\tau - \mathbf{w}_\tau) = 0.$$

We supplement this equation with the initial condition $u(0) = u_0 \in L^2(\Gamma_0)$. Now, if $\mathbf{w} = 0$, then clearly there is no evolution of the surface and $\Gamma_0 \equiv \Gamma(t)$. Let us assume for simplicity that $\mathbf{b} = \mathbf{w}$; that is, the physical velocity agrees with the velocity of the parameterisation. Let us suppose that $\Gamma(t)$ possesses the properties in §6.1.1.

Availing ourselves of the framework in §6.1.1, the weak formulation asks to find $u \in W(V, V^*)$ such that

$$\langle \dot{u}(t), v(t) \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))} + \int_{\Gamma(t)} \nabla_\Gamma u(t) \cdot \nabla_\Gamma v(t) + \int_{\Gamma(t)} u(t) v(t) \nabla_\Gamma \cdot \mathbf{w}(t) = 0$$

holds for all $v \in L_V^2$ and for almost every $t \in [0, T]$. Here,

$$a(t; u, v) = \int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma v$$

which clearly satisfy the assumptions listed in Assumptions 3.2. Applying Theorem 3.6, we obtain a unique solution $u \in W(V, V^*)$.

If instead we ask for $\dot{u} \in L^2_H$, in addition to requiring $u_0 \in H^1(\Gamma_0)$, we need to check Assumptions 3.8. We take $a_s \equiv a$ as defined above and set $a_n \equiv 0$. Most of the assumptions are easy to check. For (A6), we see from [11, Lemma 2.2] that for $\eta \in \tilde{C}^1_V$,

$$\frac{d}{dt} \int_{\Gamma(t)} |\nabla_{\Gamma} \eta(t)|^2 = \int_{\Gamma(t)} (2\nabla_{\Gamma} \eta(t) \cdot \nabla_{\Gamma} \dot{\eta}(t) - 2\nabla_{\Gamma} \eta(t) (\mathbf{D}_{\Gamma} \mathbf{w}(t)) \nabla_{\Gamma} \eta(t) + |\nabla_{\Gamma} \eta(t)|^2 \nabla_{\Gamma} \cdot \mathbf{w}(t))$$

where $(\mathbf{D}_{\Gamma} \mathbf{w}(t))_{ij} = \underline{D}_j \mathbf{w}^i(t)$. So

$$r(t; \eta) = \int_{\Gamma(t)} (-2\nabla_{\Gamma} \eta (\mathbf{D}_{\Gamma} \mathbf{w}(t)) \nabla_{\Gamma} \eta + |\nabla_{\Gamma} \eta|^2 \nabla_{\Gamma} \cdot \mathbf{w}(t))$$

which satisfies (A7). Finally, an application of Theorem 3.10 allows us to conclude well-posedness with a unique solution $u \in W(V, H)$.

Remark 6.4. We mentioned in Remark 6.2 that if \mathbf{w} is purely tangential, the surface does not evolve. However, even in this situation, it can still be useful to think of spaces of functions on $\Gamma(t) \equiv \Gamma_0$ as $H(t)$ and $V(t)$ (i.e., still parametrised by $t \in [0, T]$.) Consider the surface heat equation

$$\dot{u} - \Delta_{\Gamma} u + u \nabla_{\Gamma} \cdot \mathbf{w} = 0 \quad \text{on } \Gamma(t).$$

If $\mathbf{w}(t, \cdot)$ is a tangential velocity field, then this equation corresponds to

$$u_t - \Delta_{\Gamma} u + u \nabla_{\Gamma} \cdot \mathbf{w} + \mathbf{w} \cdot \nabla_{\Gamma} u = f \quad \text{on } \Gamma(t),$$

which could be advection-dominated (if \mathbf{w} is sufficiently large) and potentially problematic for numerical computations. The first formulation, in which we make use of $H(t)$ and $V(t)$ for each $t \in [0, T]$, avoids this issue.

6.3 Application 2: bulk equation

Let $V(t) = H^1_0(\Omega(t))$ and $H(t) = L^2(\Omega(t))$. With ϕ_t referring to the map $\phi_{\Omega, t}$ from Definition 6.3, it follows from §6.1.2 that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$ are compatible and that there is an evolving space equivalence between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$.

We consider the following boundary value problem

$$\begin{aligned} \dot{u}(t) + (\mathbf{b}(t) - \mathbf{w}(t)) \cdot \nabla u(t) + u(t) \nabla \cdot \mathbf{b}(t) - D \Delta u(t) &= f(t) && \text{on } \Omega(t) \\ u(t, \cdot) &= 0 && \text{on } \Gamma(t) \\ u(0, \cdot) &= u_0(\cdot) && \text{on } \Omega_0 \end{aligned}$$

where $D > 0$ is a constant and the physical material velocity $\mathbf{b}(t): \Omega(t) \rightarrow \mathbb{R}^n$ is such that \mathbf{b} and $\nabla \cdot \mathbf{b}$ are uniformly bounded above in time and space. We refer the reader to [8] for a formulation of

balance equations on moving time-dependent bulk domains. Our weak formulation is: with $f \in L^2_H$ and $u_0 \in V_0$, find $u \in W(V, H)$ such that

$$\begin{aligned} (\dot{u}, v)_{H(t)} + \int_{\Omega(t)} (\mathbf{b} - \mathbf{w}) \cdot \nabla uv + (\nabla \cdot \mathbf{b})uv + D\nabla u \cdot \nabla v \, dx &= \int_{\Omega(t)} f v \\ u(0) &= u_0 \end{aligned}$$

holds for all $v \in L^2_V$ and for almost every $t \in [0, T]$. For convenience, set $\mathbf{p} := \mathbf{b} - \mathbf{w}$. So we have

$$a(t; u, v) = \int_{\Omega(t)} \mathbf{p} \cdot \nabla uv + (\nabla \cdot \mathbf{b})uv + D\nabla u \cdot \nabla v$$

with the symmetric and non-symmetric parts

$$a_s(t; u, v) = \int_{\Omega(t)} D\nabla u \cdot \nabla v \quad \text{and} \quad a_n(t; u, v) = \int_{\Omega(t)} ((\nabla \cdot \mathbf{b})u + \mathbf{p} \cdot \nabla u)v$$

respectively. We need to check Assumptions 3.2 and 3.8.

Boundedness of $a(t; \cdot, \cdot)$ is easy, while coercivity can be shown by the use of Young's equality with ϵ :

$$\begin{aligned} a(t; v, v) &\geq D \|\nabla v\|_{L^2(\Omega(t))}^2 - \int_{\Omega(t)} |v \mathbf{p} \cdot \nabla v| \, dx - \|\nabla \cdot \mathbf{b}\|_{L^\infty(\Omega(t))} \|v\|_{L^2(\Omega(t))}^2 \\ &\geq D \|\nabla v\|_{L^2(\Omega(t))}^2 - \frac{C}{2D} \|\mathbf{p}^2\|_{L^\infty(\Omega(t))} \|v\|_{L^2(\Omega(t))}^2 - \frac{D}{2} \|\nabla v\|_{L^2(\Omega(t))}^2 \\ &\quad - \|\nabla \cdot \mathbf{b}\|_{L^\infty(\Omega(t))} \|v\|_{L^2(\Omega(t))}^2 \\ &= - \left(\frac{C}{2D} \|\mathbf{p}^2\|_{L^\infty(\Omega(t))} + \|\nabla \cdot \mathbf{b}\|_{L^\infty(\Omega(t))} \right) \|v\|_{L^2(\Omega(t))}^2 + \frac{D}{2} \|\nabla v\|_{L^2(\Omega(t))}^2. \end{aligned}$$

Coming to the term $a_s(t; \cdot, \cdot)$; firstly, positivity and boundedness are obvious, and differentiability is the same as for the bilinear form $a(t; \cdot, \cdot)$ in the previous example:

$$\frac{d}{dt} a_s(t; \eta(t), \eta(t)) = 2a_s(t; \dot{\eta}(t), \eta(t)) + r(t; \eta(t))$$

for $\eta \in \tilde{C}_V^1$, where

$$r(t; \eta(t)) = D \int_{\Omega(t)} (-2\nabla \eta(t) (\mathbf{D}_\Omega \mathbf{w}(t)) \nabla \eta(t) + |\nabla \eta(t)|^2 \nabla \cdot \mathbf{w}(t))$$

which is obviously bounded. Finally, the uniform bound on $a_n(t; \cdot, \cdot): V(t) \times H(t) \rightarrow \mathbb{R}$ is easy to see.

With all the assumptions checked, we apply Theorem 3.10 and find a unique solution $u \in W(V, H)$.

6.4 Application 3: coupled bulk-surface system

In [14], the authors consider well-posedness of an elliptic coupled bulk-surface system on a static domain; we now extend this to the parabolic case on an evolving domain. We want to find solutions $u(t): \Omega(t) \rightarrow \mathbb{R}$ and $v(t): \Gamma(t) \rightarrow \mathbb{R}$ of the coupled bulk-surface PDE system

$$\dot{u} - \Delta_{\Omega}u + u\nabla_{\Omega} \cdot \mathbf{w} = f \quad \text{on } \Omega(t) \quad (6.5)$$

$$\dot{v} - \Delta_{\Gamma}v + v\nabla_{\Gamma} \cdot \mathbf{w} + \nabla_{\Omega}u \cdot \nu = g \quad \text{on } \Gamma(t) \quad (6.6)$$

$$\nabla_{\Omega}u \cdot \nu = \beta v - \alpha u \quad \text{on } \Gamma(t) \quad (6.7)$$

$$u(0) = u_0 \quad \text{on } \Omega_0 \quad (6.8)$$

$$v(0) = v_0 \quad \text{on } \Gamma_0 \quad (6.9)$$

where $\alpha, \beta > 0$ are constants. Note that (6.7) is a Robin boundary condition for $\Omega(t)$ and that we reused the notation u for denoting the trace of u . Note that we assume there is just the one velocity field \mathbf{w} .)

6.4.1 Function spaces

Define the product Hilbert spaces

$$V(t) = H^1(\Omega(t)) \times H^1(\Gamma(t)) \quad \text{and} \quad H(t) = L^2(\Omega(t)) \times L^2(\Gamma(t))$$

which we equip with the inner products

$$((\omega_1, \gamma_1), (\omega_2, \gamma_2))_{H(t)} = (\omega_1, \omega_2)_{L^2(\Omega(t))} + (\gamma_1, \gamma_2)_{L^2(\Gamma(t))}$$

$$((\omega_1, \gamma_1), (\omega_2, \gamma_2))_{V(t)} = (\omega_1, \omega_2)_{H^1(\Omega(t))} + (\gamma_1, \gamma_2)_{H^1(\Gamma(t))}.$$

Clearly $V(t) \subset H(t)$ is continuous and dense and both spaces are separable. The dual space of $V(t)$ is

$$V^*(t) = H^{-1}(\Omega(t)) \times H^{-1}(\Gamma(t))$$

and the duality pairing is

$$\langle (f_{\omega}, f_{\gamma}), (u_{\omega}, u_{\gamma}) \rangle_{V^*(t), V(t)} = \langle f_{\omega}, u_{\omega} \rangle_{H^{-1}(\Omega(t)), H^1(\Omega(t))} + \langle f_{\gamma}, u_{\gamma} \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))}.$$

Define the map

$$\phi_t: H_0 \rightarrow H(t)$$

by

$$\phi_t((\omega, \gamma)) = (\phi_{\Omega, t}(\omega), \phi_{\Gamma, t}(\gamma))$$

where $\phi_{\Omega, t}$ and $\phi_{\Gamma, t}$ are as defined previously. From §6.1.1 and §6.1.2, we find that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$ are compatible, and we have the evolving space equivalence between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$.

To define the weak material derivative, note that because the inner product on $H(t)$ is a sum of the L^2 inner products on $\Omega(t)$ and $\Gamma(t)$, it follows that the bilinear form $c(t; \cdot, \cdot)$ is

$$c(t; (\omega_1, \gamma_1), (\omega_2, \gamma_2)) = c_\Omega(t; \omega_1, \omega_2) + c_\Gamma(t; \gamma_1, \gamma_2)$$

with

$$c_\Omega(t; \omega_1, \omega_2) = \int_{\Omega(t)} \omega_1 \omega_2 \nabla_\Omega \cdot \mathbf{w}(t) \quad \text{and} \quad c_\Gamma(t; \gamma_1, \gamma_2) = \int_{\Gamma(t)} \gamma_1 \gamma_2 \nabla_\Gamma \cdot \mathbf{w}(t)$$

being the bilinear forms (which we called $c(t; \cdot, \cdot)$) associated with the material derivatives of the constituent spaces of the product space.

6.4.2 Weak formulation and well-posedness

To obtain the weak form, we let $(\omega, \gamma) \in L_V^2$ and take the inner product of (6.5) with ω and the inner product of (6.6) with γ :

$$\int_{\Omega(t)} \dot{u}\omega + \int_{\Omega(t)} \nabla_\Omega u \cdot \nabla_\Omega \omega - \int_{\Gamma(t)} \omega \nabla_\Omega u \cdot \nu + \int_{\Omega(t)} u\omega \nabla_\Omega \cdot \mathbf{w} = \int_{\Omega(t)} f\omega \quad (6.10)$$

$$\int_{\Gamma(t)} \dot{v}\gamma + \int_{\Gamma(t)} \nabla_\Gamma v \cdot \nabla_\Gamma \gamma + \int_{\Gamma(t)} v\gamma \nabla_\Gamma \cdot \mathbf{w} + \int_{\Gamma(t)} \gamma \nabla_\Omega u \cdot \nu = \int_{\Gamma(t)} g\gamma. \quad (6.11)$$

Multiplying (6.10) by α and (6.11) by β , taking the sum and substituting the boundary condition (6.7), we end up with

$$\begin{aligned} & \alpha \int_{\Omega(t)} \dot{u}\omega + \beta \int_{\Gamma(t)} \dot{v}\gamma + \alpha \int_{\Omega(t)} \nabla_\Omega u \cdot \nabla_\Omega \omega + \beta \int_{\Gamma(t)} \nabla_\Gamma v \cdot \nabla_\Gamma \gamma + \alpha \int_{\Omega(t)} u\omega \nabla_\Omega \cdot \mathbf{w} \\ & + \beta \int_{\Gamma(t)} v\gamma \nabla_\Gamma \cdot \mathbf{w} + \int_{\Gamma(t)} (\beta v - \alpha u)(\beta \gamma - \alpha \omega) = \alpha \int_{\Omega(t)} f\omega + \beta \int_{\Gamma(t)} g\gamma. \end{aligned}$$

Defining the bilinear forms

$$\begin{aligned} l(t; (\dot{u}, \dot{v}), (\omega, \gamma)) &= \alpha \langle \dot{u}, \omega \rangle_{H^{-1}(\Omega(t)), H^1(\Omega(t))} + \beta \langle \dot{v}, \gamma \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))} \\ a(t; (u, v), (\omega, \gamma)) &= \alpha \int_{\Omega(t)} \nabla_\Omega u \cdot \nabla_\Omega \omega + \beta \int_{\Gamma(t)} \nabla_\Gamma v \cdot \nabla_\Gamma \gamma + \int_{\Gamma(t)} (\beta v - \alpha u)(\beta \gamma - \alpha \omega), \end{aligned}$$

our weak formulation reads: given $(f, g) \in L_H^2$ and $(u_0, v_0) \in V_0$, find $(u, v) \in W(V, V^*)$ such that

$$\begin{aligned} l(t; (\dot{u}, \dot{v}), (\omega, \gamma)) + a(t; (u, v), (\omega, \gamma)) + c(t; (u, v), (\omega, \gamma)) &= ((\alpha f, \alpha g), (\omega, \gamma))_{H(t)} \\ (u(0), v(0)) &= (u_0, v_0) \end{aligned} \quad (\mathbf{P}_{\text{bs}})$$

for all $(\omega, \gamma) \in L_V^2$ and for almost every $t \in [0, T]$.

Let us now check Assumptions 3.1.

Assumptions (L1)–(L8) We can write

$$l(t; (\dot{u}, \dot{v}), (\omega, \gamma)) = \langle \mathcal{L}(t)(\dot{u}, \dot{v}), (\omega, \gamma) \rangle_{V^*(t), V(t)} = \langle (\alpha \dot{u}, \beta \dot{v}), (\omega, \gamma) \rangle_{V^*(t), V(t)},$$

i.e., $\mathcal{L}(t)(\dot{u}, \dot{v})$ is the functional $\langle (\alpha \dot{u}, \beta \dot{v}), \cdot \rangle_{V^*(t), V(t)}$, which obviously satisfies (L1). When $(\dot{u}, \dot{v}) \in H(t)$,

$$\langle \mathcal{L}(t)(\dot{u}, \dot{v}), (\omega, \gamma) \rangle = ((\alpha \dot{u}, \beta \dot{v}), (\omega, \gamma))_{H(t)},$$

so indeed $\mathcal{L}(t)|_{H(t)}$ has range in $H(t)$ and $\mathcal{L}(t)|_{V(t)}$ has range in $V(t)$. Assumptions (L2)–(L5) are immediate, and (L6) also follows easily. For (L7) and (L8), note that the map $\dot{\mathcal{L}} \equiv 0$.

We also need to check Assumptions 3.2 and 3.8 on the bilinear form $a(t; \cdot, \cdot)$. We shall use the trace inequality [35, §I.8, Theorem 8.7, p. 126]:

$$\|\tau_t u\|_{L^2(\Gamma(t))} \leq C_T \|u\|_{H^1(\Omega(t))} \quad \text{for all } u \in H^1(\Omega(t)),$$

and we assume that the constant is independent of $t \in [0, T]$. Set $\mathbf{v}_i = (\omega_i, \gamma_i)$ for $i = 1, 2$. Coercivity of $a(t; \cdot, \cdot)$ (assumption (A1)) is achieved with no great difficulty (one uses the L^∞ bound on $\mathbf{w} \cdot \boldsymbol{\mu}$, the trace inequality and Young's inequality with ϵ).

Assumption (A2) For boundedness of $a(t; \cdot, \cdot)$, we start with

$$|a(t; \mathbf{v}_1, \mathbf{v}_2)| \leq C \|\mathbf{v}_1\|_{V(t)} \|\mathbf{v}_2\|_{V(t)} + \int_{\Gamma(t)} |\beta \gamma_1 \gamma_2 + \alpha \omega_1 \omega_2 - \alpha \omega_1 \gamma_2 - \beta \gamma_1 \omega_2|. \quad (6.12)$$

The trace inequality allows us to estimate the last term of (6.12) as follows:

$$\begin{aligned} & \int_{\Gamma(t)} |\beta \gamma_1 \gamma_2 + \alpha \omega_1 \omega_2 - \alpha \omega_1 \gamma_2 - \beta \gamma_1 \omega_2| \\ & \leq \beta \|\gamma_1\|_{L^2(\Gamma(t))} \|\gamma_2\|_{L^2(\Gamma(t))} + \alpha C_T^2 \|\omega_1\|_{H^1(\Omega(t))} \|\omega_2\|_{H^1(\Omega(t))} \\ & \quad + \alpha C_T \|\omega_1\|_{H^1(\Omega(t))} \|\gamma_2\|_{L^2(\Gamma(t))} + \beta C_T \|\gamma_1\|_{L^2(\Gamma(t))} \|\omega_2\|_{H^1(\Omega(t))} \\ & \leq C \|(\omega_1, \gamma_1)\|_{V(t)} \|(\omega_2, \gamma_2)\|_{V(t)} = C \|\mathbf{v}_1\|_{V(t)} \|\mathbf{v}_2\|_{V(t)}. \end{aligned}$$

Assumptions (A6) and (A7) We do not require the splitting of $a(t; \cdot, \cdot)$ into a differentiable and non-differentiable part, since $a(t; \cdot, \cdot)$ is differentiable as shown below. In view of this and Remark 3.9, we need only to check (A6) and (A7). Let us define

$$\begin{aligned} a_\Omega(t; \omega_1, \omega_2) &= \alpha \int_{\Omega(t)} \nabla_\Omega \omega_1 \cdot \nabla_\Omega \omega_2 \\ a_\Gamma(t; \gamma_1, \gamma_2) &= \beta \int_{\Gamma(t)} \nabla_\Gamma \gamma_1 \cdot \nabla_\Gamma \gamma_2 \end{aligned}$$

so that

$$a(t; (\omega_1, \gamma_1), (\omega_2, \gamma_2)) = a_\Omega(t; \omega_1, \omega_2) + a_\Gamma(t; \gamma_1, \gamma_2) + \int_{\Gamma(t)} (\beta \gamma_1 - \alpha \omega_1)(\beta \gamma_2 - \alpha \omega_2)$$

Taking $\mathbf{v}_1 \in \tilde{C}_V^1$, we differentiate:

$$\begin{aligned} \frac{d}{dt}a(t; \mathbf{v}_1, \mathbf{v}_1) &= 2a_\Omega(t; \dot{\omega}_1, \omega_1) + r_\Omega(t; \omega_1) + 2a_\Gamma(t; \dot{\gamma}_1, \gamma_1) + r_\Gamma(t; \gamma_1) \\ &\quad + 2(\beta\dot{\gamma}_1 - \alpha\dot{\omega}_1, \beta\gamma_1 - \alpha\omega_1)_{L^2(\Gamma(t))} + c_\Gamma(t; \beta\gamma_1 - \alpha\omega_1, \beta\gamma_1 - \alpha\omega_1) \\ &= 2a(t; (\dot{\omega}_1, \dot{\gamma}_1), (\omega_1, \gamma_1)) + r(t; (\omega_1, \gamma_1)) \\ &= 2a(t; \dot{\mathbf{v}}_1, \mathbf{v}_1) + r(t; \mathbf{v}_1). \end{aligned}$$

Here, we defined

$$r(t; (\omega_1, \gamma_1)) = r_\Omega(t; \omega_1) + r_\Gamma(t; \gamma_1) + c_\Gamma(t; \beta\gamma_1 - \alpha\omega_1, \beta\gamma_1 - \alpha\omega_1)$$

where r_Ω is the form r from §6.2 with domain Ω and r_Γ is (also) the form r from §6.2 with domain Γ . By the bounds on r_Ω , r_Γ and c , we have

$$\begin{aligned} |r(t; \mathbf{v}_1)| &\leq C_1(\|\omega_1\|_{H^1(\Omega(t))}^2 + \|\gamma_1\|_{H^1(\Gamma(t))}^2 + \|\beta\gamma_1 - \alpha\omega_1\|_{L^2(\Gamma(t))}^2) \\ &\leq C_2(\|\omega_1\|_{H^1(\Omega(t))}^2 + \|\gamma_1\|_{H^1(\Gamma(t))}^2 + \|\gamma_1\|_{L^2(\Gamma(t))}^2 + \|\omega_1\|_{L^2(\Gamma(t))}^2) \\ &\leq C_2((1 + C_T^2) \|\omega_1\|_{H^1(\Omega(t))}^2 + 2\|\gamma_1\|_{H^1(\Gamma(t))}^2) \\ &\leq C_3 \|\mathbf{v}_1\|_{V(t)}^2, \end{aligned}$$

i.e. $r(t; \cdot)$ is bounded in $V(t)$.

With all the assumptions satisfied, we find from Theorem 3.10 that there is a unique solution $(u, v) \in W(V, H)$ to the problem (\mathbf{P}_{bs}) .

6.5 Application 4: dynamic boundary condition for elliptic equation on moving domain

Given $f \in L^2_{H^{-\frac{1}{2}}}$ and $w_0 \in L^2(\Gamma_0)$, we consider the problem of finding a function $w(t) : \Omega(t) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \Delta w(t) &= 0 \quad \text{on } \Omega(t) \\ \dot{w}(t) + \frac{\partial w(t)}{\partial \nu} + w(t) &= f(t) \quad \text{on } \Gamma(t) \\ w(0) &= w_0 \quad \text{on } \Gamma_0 \end{aligned} \tag{6.13}$$

is satisfied in a weak sense. This is a natural extension to evolving domains of the problem considered in §1.11.1 of [23].

6.5.1 Function spaces

We assume some stronger regularity on the map Φ_t^0 here, namely

$$\begin{aligned} \Phi_t^0 : \Gamma_0 &\rightarrow \Gamma(t) \text{ is a } C^3\text{-diffeomorphism, and} \\ \Phi_{(\cdot)}^0 &\in C^3([0, T] \times \Gamma_0). \end{aligned}$$

In this case, we use the pivot space $H(t) = L^2(\Gamma(t))$ but now require $V(t) = H^{\frac{1}{2}}(\Gamma(t))$. Below, we shall mainly make use of $\phi_{\Gamma,t}$ and to save space we shall write it simply as ϕ_t . We only revert to the full notation when ambiguity forces us to.

We use the notation and the established results of §6.1.1, from which we already know that $\phi_{-t}: L^2(\Gamma(t)) \rightarrow L^2(\Gamma_0)$ is a well-defined linear homeomorphism. Now we show that the map $\phi_{-t}: H^{\frac{1}{2}}(\Gamma(t)) \rightarrow H^{\frac{1}{2}}(\Gamma_0)$ is also a linear homeomorphism. Let $u \in H^{\frac{1}{2}}(\Gamma(t))$. It suffices to estimate only the seminorm $|\phi_{-t}u|_{H^{\frac{1}{2}}(\Gamma_0)}$. We have

$$\int_{\Gamma_0} \int_{\Gamma_0} \frac{|\phi_{-t}u(x) - \phi_{-t}u(y)|^2}{|x - y|^n} = \int_{\Gamma(t)} \int_{\Gamma(t)} \frac{|u(x_t) - u(y_t)|^2}{|\Phi_0^t(x_t) - \Phi_0^t(y_t)|^n} J_0^t(x_t) J_0^t(y_t) \quad (6.14)$$

where we made the substitutions $x_t = \Phi_t^0(x) \in \Gamma(t)$ and $y_t = \Phi_t^0(y) \in \Gamma(t)$. Since Φ_t^0 is a C^1 -diffeomorphism between compact spaces, it is bi-Lipschitz with Lipschitz constant C_L independent of t (because the spatial derivatives of Φ_t^0 are uniformly bounded). This implies

$$|x_t - y_t| \leq C_L |\Phi_0^t(x_t) - \Phi_0^t(y_t)|$$

so that (6.14) becomes

$$|\phi_{-t}u|_{H^{\frac{1}{2}}(\Gamma_0)}^2 \leq C_L^n C_J^2 \int_{\Gamma(t)} \int_{\Gamma(t)} \frac{|u(x_t) - u(y_t)|^2}{|x_t - y_t|^n} = C_L^n C_J^2 |u|_{H^{\frac{1}{2}}(\Gamma(t))}^2,$$

where we used the uniform bound on J_0^t . So we have the uniform bound

$$\|\phi_{-t}u\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C \|u\|_{H^{\frac{1}{2}}(\Gamma(t))}.$$

A similar bound holds for the operator ϕ_t by the same arguments as above since $\Phi_0^t = (\Phi_t^0)^{-1}$ also satisfies the same properties as above.

It follows by the smoothness on $\Phi_{(\cdot)}^0$ that $J_{(\cdot)}^0 \in C^2([0, T] \times \Gamma_0)$. This implies that $J_t^0: \Gamma_0 \rightarrow \mathbb{R}$ is (globally) Lipschitz (see the paragraph after the proof of Proposition 2.4 in [21, p. 23]).

The map

$$\begin{aligned} t \mapsto |\phi_t u|_{H^{\frac{1}{2}}(\Gamma(t))}^2 &= \int_{\Gamma(t)} \int_{\Gamma(t)} \frac{|\phi_t u(x) - \phi_t u(y)|^2}{|x - y|^n} \\ &= \int_{\Gamma_0} \int_{\Gamma_0} \frac{|u(x_0) - u(y_0)|^2}{|\Phi_t^0(x_0) - \Phi_t^0(y_0)|^n} J_t^0(x_0) J_t^0(y_0) \end{aligned}$$

is continuous. To see this, define the integrand

$$g(x_0, y_0, t) = \frac{|u(x_0) - u(y_0)|^2}{|\Phi_t^0(x_0) - \Phi_t^0(y_0)|^n} J_t^0(x_0) J_t^0(y_0).$$

Now, $t \mapsto g(x_0, y_0, t)$ is continuous for almost all (x_0, y_0) (it only fails when the denominator is zero, where $x_0 = y_0$, and the set of such points has zero measure), and we have the domination

$g(x_0, y_0, t) \leq h(x_0, y_0)$ for all t and almost all (x_0, y_0) by an integrable function h ; this follows due to the smoothness assumptions on $\Phi_{(\cdot)}^0$ and $J_{(\cdot)}^0$. Therefore, $t \mapsto \int_{\Gamma_0} \int_{\Gamma_0} g(x_0, y_0, t)$ is continuous. This enables to conclude that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$ are compatible.

There is some effort needed in order to show the evolving space equivalence. We start with the following two results which are used continually.

Lemma 6.5. For $y \in \Gamma_0$, we have

$$\int_{\Gamma_0} \frac{1}{|x - y|^{n-2}} d\sigma(x) < C$$

where C does not depend on y .

This lemma can be proved by first setting $y = 0$ (without loss of generality) and then splitting the domain of integration into two terms, one of which is a ball centred at the the origin. The integral over the ball can be tackled with the assumption of the domain being Lipschitz and switching to polar coordinates, while the integral over the complement of the ball is obviously finite.

Lemma 6.6. If $\rho \in C^1(\Gamma_0)$ and $u \in H^{\frac{1}{2}}(\Gamma_0)$ then $\rho u \in H^{\frac{1}{2}}(\Gamma_0)$ and

$$\|\rho u\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C \|\rho\|_{C^1(\Gamma_0)} \|u\|_{H^{\frac{1}{2}}(\Gamma_0)} \quad (6.15)$$

where C does not depend on ρ or u .

Proof. Note that ρ and $\nabla\rho$ are bounded from above and ρ is Lipschitz. We begin with

$$\|\rho u\|_{H^{\frac{1}{2}}(\Gamma_0)}^2 \leq \|\rho\|_{C^0(\Gamma_0)}^2 \|u\|_{L^2(\Gamma_0)}^2 + \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\rho(x)u(x) - \rho(y)u(y)|^2}{|x - y|^n} dx dy.$$

The last term is

$$\begin{aligned} & \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\rho(x)u(x) - \rho(y)u(y)|^2}{|x - y|^n} \\ & \leq 2 \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\rho(x)|^2 |u(x) - u(y)|^2}{|x - y|^n} + 2 \int_{\Gamma_0} \int_{\Gamma_0} \frac{|u(y)|^2 |\rho(x) - \rho(y)|^2}{|x - y|^n} \\ & \leq 2 \|\rho\|_{C^0(\Gamma_0)}^2 \|u\|_{H^{\frac{1}{2}}(\Gamma_0)}^2 + 2 \|\nabla\rho\|_{C^0(\Gamma_0)}^2 \int_{\Gamma_0} \int_{\Gamma_0} \frac{|u(y)|^2}{|x - y|^{n-2}}. \end{aligned}$$

From the previous lemma, the integral in the second term is

$$\int_{\Gamma_0} \int_{\Gamma_0} \frac{|u(y)|^2}{|x - y|^{n-2}} = \int_{\Gamma_0} |u(y)|^2 \int_{\Gamma_0} |x - y|^{2-n} \leq C_1 \|u\|_{L^2(\Gamma_0)}^2.$$

Putting it all together, we achieve (6.15). \square

In the following lemmas, let $J \in C^2([0, T] \times \Gamma_0)$.

Lemma 6.7. If $\psi \in \mathcal{D}((0, T); H^{\frac{1}{2}}(\Gamma_0))$, then $\psi J \in W(V_0, V_0^*)$ and $(\psi J)' = \psi' J + \psi J'$.

Proof. Let us note that

$$\begin{aligned}\psi &\in C^0([0, T]; H^{\frac{1}{2}}(\Gamma_0)) \\ J &\in C^0([0, T]; H^{\frac{1}{2}}(\Gamma_0)).\end{aligned}$$

The second inclusion holds because $J \in C^0([0, T]; H^1(\Gamma_0))$ by [34, Lemma 3.6], and because $H^1(\Gamma_0) \subset H^{\frac{1}{2}}(\Gamma_0)$ is continuous [32, Theorem 2.5.1 and Theorem 2.5.5].

Now, note that $\psi(t)J(t) \in H^{\frac{1}{2}}(\Gamma_0)$ for all t by Lemma 6.6. To see that $\psi J \in C^0([0, T]; H^{\frac{1}{2}}(\Gamma))$, let $t_n \rightarrow t$ and consider

$$\begin{aligned}\|\psi(t)J(t) - \psi(t_n)J(t_n)\|_{H^{\frac{1}{2}}(\Gamma_0)} &\leq \|\psi(t)(J(t) - J(t_n))\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|J(t_n)(\psi(t) - \psi(t_n))\|_{H^{\frac{1}{2}}(\Gamma_0)} \\ &\leq C\|J(t) - J(t_n)\|_{C^1(\Gamma_0)} \|\psi(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} \\ &\quad + \|J(t_n)\|_{C^1(\Gamma_0)} \|\psi(t) - \psi(t_n)\|_{H^{\frac{1}{2}}(\Gamma_0)}.\end{aligned}$$

The first of these terms tends to zero as $t_n \rightarrow t$ because $J \in C^0([0, T]; C^1(\Gamma_0))$ [34, Lemma 3.6], and the second because $\psi \in C^1([0, T]; H^{\frac{1}{2}}(\Gamma_0))$.

Now we show that in fact $\psi J \in C^1([0, T]; H^{\frac{1}{2}}(\Gamma_0))$ and that $(\psi J)' = \psi' J + \psi J'$. Observe that $\psi'(t)J(t) + \psi(t)J'(t) \in H^{\frac{1}{2}}(\Gamma_0)$ by Lemma 6.6. Define the difference quotients $D^h J(t) = (J(t+h) - J(t))/h$ and $D^h \psi(t)$ similarly and note that

$$\begin{aligned}&\left\| \frac{\psi(t+h)J(t+h) - \psi(t)J(t)}{h} - \psi'(t)J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\ &\leq \|\psi(t+h)D^h J(t) - \psi(t)J'(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|D^h \psi(t)J(t) - \psi'(t)J(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} \\ &\leq C\|D^h J(t) - J'(t)\|_{C^1(\Gamma_0)} \|\psi(t+h)\|_{H^{\frac{1}{2}}(\Gamma_0)} + C\|J'(t)\|_{C^1(\Gamma_0)} \|\psi(t+h) - \psi(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} \\ &\quad + \|J(t)\|_{C^1(\Gamma_0)} \|D^h \psi(t) - \psi'(t)\|_{H^{\frac{1}{2}}(\Gamma_0)}\end{aligned}$$

because both $J, \nabla J \in C^1([0, T]; C^0(\Gamma_0))$ [34, Lemma 3.6] (which we can use because $J \in C^2([0, T] \times \Gamma_0)$). Note that here we used

$$\begin{aligned}\|\psi(t+h)D^h J(t) - \psi(t)J'(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} &\leq \|\psi(t+h)(D^h J(t) - J'(t))\|_{H^{\frac{1}{2}}(\Gamma_0)} \\ &\quad + \|(\psi(t+h) - \psi(t))J'(t)\|_{H^{\frac{1}{2}}(\Gamma_0)}.\end{aligned}$$

Thus, we find

$$\lim_{h \rightarrow 0} \left\| \frac{\psi(t+h)J(t+h) - \psi(t)J(t)}{h} - \psi'(t)J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} = 0. \quad \square$$

Theorem 6.8. For every $u \in \mathcal{W}(V_0, V_0^*)$, $Ju \in \mathcal{W}(V_0, V_0^*)$.

Proof. Let $\psi \in \mathcal{D}((0, T); H^{\frac{1}{2}}(\Gamma_0))$ and for $u \in W(V_0, V_0^*)$, consider

$$\begin{aligned} \int_0^T \langle u'(t), J(t)\psi(t) \rangle_{H^{-\frac{1}{2}}(\Gamma_0), H^{\frac{1}{2}}(\Gamma_0)} &= - \int_0^T (J'(t)\psi(t) + J(t)\psi'(t), u(t))_{L^2(\Gamma_0)} \\ &\quad \text{(by the formula of partial integration and the last lemma)} \\ &= - \int_0^T (\psi(t), J'(t)u(t))_{L^2(\Gamma_0)} - \int_0^T (\psi'(t), J(t)u(t))_{L^2(\Gamma_0)}. \end{aligned}$$

Rearranging yields

$$\int_0^T (J(t)u(t), \psi'(t))_{L^2(\Gamma_0)} = - \int_0^T \langle J'(t)u(t) + J(t)u'(t), \psi(t) \rangle_{H^{-\frac{1}{2}}(\Gamma_0), H^{\frac{1}{2}}(\Gamma_0)}.$$

This shows that Ju has a weak derivative, and $(Ju)' \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma_0))$ since we have $J'u \in L^2(0, T; H^{\frac{1}{2}}(\Gamma_0))$ and $Ju' \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma_0))$. \square

Theorem 6.9. We have the evolving space equivalence between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$.

Proof. The last result shows that if $u \in \mathcal{W}(V_0, V_0^*)$ then $J_t^0 u \in \mathcal{W}(V_0, V_0^*)$. Because $1/J_t^0 \in C^2([0, T] \times \overline{\Gamma_0})$, the converse also holds (using $J = 1/J_t^0$). Since

$$(J_t^0 u(t))' = J_t^0 u'(t) + \hat{C}(t)u(t),$$

we have $\hat{S}(t) = T_t = J_t^0$ and $\hat{D}(t) \equiv 0$, and it follows that $\hat{S}(\cdot)u'(\cdot) \in L^2(0, T; H^{-\frac{1}{2}}(\Gamma_0))$. Thus the equivalence of norms is also true due to Theorem 2.32. \square

6.5.2 Well-posedness

We first need the following auxiliary result.

Lemma 6.10. Given $u \in H^{\frac{1}{2}}(\Gamma(t))$, there exists a unique weak solution $\tilde{w} \in H^1(\Omega(t))$ to

$$\begin{aligned} \Delta \tilde{w} &= 0 \quad \text{on } \Omega(t) \\ \tilde{w} &= \tilde{u} \quad \text{on } \Gamma(t). \end{aligned} \tag{6.16}$$

that satisfies

$$\|\tilde{w}\|_{H^1(\Omega(t))} \leq C \|\tilde{u}\|_{H^{\frac{1}{2}}(\Gamma(t))}$$

where the constant C does not depend on $t \in [0, T]$.

The existence and uniqueness of the solution to (6.16) is well-studied but the continuous dependence with the constant independent of $t \in [0, T]$ requires a proof. For that, we need the following results which show that certain standard results are in a sense uniform in $t \in [0, T]$.

Lemma 6.11. Let $\tau_t: H^1(\Omega(t)) \rightarrow H^{\frac{1}{2}}(\Gamma(t))$ denote the trace map. The equality in $H^{\frac{1}{2}}(\Gamma(t))$

$$\tau_t(\phi_{\Omega, t} w) = \phi_{\Gamma, t}(\tau_0 w) \quad \text{for all } w \in H^1(\Omega_0).$$

holds.

Proof. This is because

$$\tau_t(\phi_{\Omega,t}w_n) = \phi_{\Gamma,t}(\tau_0w_n)$$

for all $w_n \in C^1(\overline{\Omega_0})$ (one can see this identity by using the facts that the same formula defines $\phi_{\Omega,t}$ and $\phi_{\Gamma,t}$ and that Φ_0^t maps boundary to boundary), in particular, it holds for $w_n \in C^1(\overline{\Omega_0}) \cap H^1(\Omega_0)$ such that $w_n \rightarrow w$ in $H^1(\Omega_0)$. Then by continuity of the various maps, we can pass to the limit and obtain the identity. \square

Lemma 6.12. For each $t \in [0, T]$, we have

$$\|v\|_{H^1(\Omega(t))} \leq C_1 \|\nabla v\|_{L^2(\Omega(t))} \quad \forall v \in H_0^1(\Omega(t)) \quad (6.17)$$

$$\|\nabla v\|_{L^2(\Omega(t))}^2 + \|v\|_{L^2(\Gamma(t))}^2 \geq C_2 \|v\|_{H^1(\Omega(t))}^2 \quad \forall v \in H^1(\Omega(t)) \quad (6.18)$$

$$\inf_{v \in H^1(\Omega(t)), \tau_t v = u} \|v\|_{H^1(\Omega(t))} \leq C_3 \|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \quad \forall u \in H^{\frac{1}{2}}(\Gamma(t)) \quad (6.19)$$

where C_1 , C_2 , and C_3 do not depend on t .

The strategy is to start with each respective inequality at $t = 0$; (6.17) is the Poincaré inequality on Ω_0 , (6.18) follows by a compactness argument and (6.19) is an equivalence of norms. Then for (6.17), use the chain rule $\nabla(\phi_{-t}v) = \nabla(v(\Phi_t^0)) = \phi_{-t}(\nabla v)\mathbf{D}\Phi_t^0$ and the uniform boundedness of $\mathbf{D}\Phi_t^0$. The inequality (6.18) is obtained with the identity $\nabla v = \nabla(\phi_{-t}\phi_t v) = \phi_{-t}(\nabla\phi_t v)\mathbf{D}\Phi_t^0$ and Lemma 6.11. The lemma is also the key ingredient to show (6.19).

Proof of Lemma 6.10. First, we use the trace map $\tau_t: H^1(\Omega(t)) \rightarrow H^{\frac{1}{2}}(\Gamma(t))$ to see that there is a function $\tilde{w}_{\tilde{u}} \in H^1(\Omega(t))$ such that $\tau_t \tilde{w}_{\tilde{u}} = \tilde{u}$. Set $d := \tilde{w} - \tilde{w}_{\tilde{u}}$. Then d solves

$$\begin{aligned} \Delta d &= -\Delta \tilde{w}_{\tilde{u}} && \text{on } \Omega(t) \\ d &= 0 && \text{on } \Gamma(t). \end{aligned} \quad (6.20)$$

Define $b_t(\cdot, \cdot): H^1(\Omega(t)) \times H^1(\Omega(t)) \rightarrow \mathbb{R}$ and $l_t(\cdot): H^1(\Omega(t)) \rightarrow \mathbb{R}$ by

$$b_t(d, \varphi) = \int_{\Omega(t)} \nabla d \nabla \varphi \quad \text{and} \quad l_t(\varphi) = \int_{\Omega(t)} \nabla \tilde{w}_{\tilde{u}} \nabla \varphi.$$

Clearly l_t and b_t are bounded and the Poincaré inequality (6.17) implies that b_t is coercive with coercivity constant C_P independent of t . By Lax–Milgram, there is a unique solution $d \in H_0^1(\Omega(t))$ to (6.20) satisfying

$$\|d\|_{H^1(\Omega(t))} \leq C_P \|\tilde{w}_{\tilde{u}}\|_{H^1(\Omega(t))}.$$

Because this inequality holds for all lifts $\tilde{w}_{\tilde{u}}$ of \tilde{u} we must have

$$\begin{aligned} \|d\|_{H^1(\Omega(t))} &\leq C_P \inf_{v \in H^1(\Omega(t)), \tau_t v = \tilde{u}} \|v\|_{H^1(\Omega(t))} \\ &\leq C_1 \|\tilde{u}\|_{H^{\frac{1}{2}}(\Gamma(t))} \end{aligned}$$

where the second inequality is thanks to (6.19). Since $\tilde{w} = d + \tilde{w}_{\tilde{u}}$, we see that (6.16) has a unique solution $\tilde{w} \in H^1(\Omega(t))$ with

$$\|\tilde{w}\|_{H^1(\Omega(t))} \leq C_2 \|\tilde{u}\|_{H^{\frac{1}{2}}(\Gamma(t))}. \quad \square$$

We write the solution of (6.16) as $w = \mathbb{D}(t)u$ where $\mathbb{D}(t): H^{\frac{1}{2}}(\Gamma(t)) \rightarrow H^1(\Omega(t))$ is the solution map which we proved is uniformly bounded

$$\|\mathbb{D}(t)u\|_{H^1(\Omega(t))} \leq C \|u\|_{H^{\frac{1}{2}}(\Gamma(t))}. \quad (6.21)$$

Back to the equation (6.13), let us note that $w(t)$ has a normal derivative (see [20, Theorem 1.5.1.2]), and we can define the **Dirichlet-to-Neumann map** $\mathbb{A}(t): H^{\frac{1}{2}}(\Gamma(t)) \rightarrow H^{-\frac{1}{2}}(\Gamma(t))$ (which is also bounded) by

$$\mathbb{A}(t)u(t) = \frac{\partial w(t)}{\partial \nu}.$$

So by a solution of the PDE (6.13), we seek a function $w(t) = \mathbb{D}(t)u(t) \in H^1(\Omega(t))$ (so $\Delta_t w(t) = 0$ and $w(t)|_{\Gamma(t)} = u(t)$) where $u(t) \in H^{\frac{1}{2}}(\Omega(t))$ satisfies

$$\begin{aligned} \dot{u}(t) + \mathbb{A}(t)u(t) + u(t) &= f(t) \quad \text{on } \Gamma(t) \\ u(0) &= w_0 \quad \text{on } \Gamma_0. \end{aligned} \quad (6.22)$$

Let $V(t) = H^{\frac{1}{2}}(\Gamma(t))$ and $H(t) = L^2(\Gamma(t))$. With $v \in L^2_V$ and using (6.1),

$$\langle \mathbb{A}(t)u(t), v(t) \rangle_{H^{-\frac{1}{2}}(\Gamma(t)), H^{\frac{1}{2}}(\Gamma(t))} = \int_{\Omega} \nabla(\mathbb{D}(t)u(t)) \nabla(\mathbb{E}(t)v(t)).$$

So the bilinear form $a(t; \cdot, \cdot): H^{\frac{1}{2}}(\Gamma(t)) \times H^{\frac{1}{2}}(\Gamma(t)) \rightarrow \mathbb{R}$ is

$$a(t; u, v) := \int_{\Omega(t)} \nabla(\mathbb{D}(t)u) \nabla(\mathbb{E}(t)v) + \int_{\Gamma(t)} uv.$$

We take $E = \mathbb{D}$, and we obtain by the uniform bound (6.21) the boundedness of $a(t; \cdot, \cdot)$:

$$\begin{aligned} |a(t; u, v)| &\leq \|\mathbb{D}(t)u\|_{H^1(\Omega(t))} \|\mathbb{D}(t)v\|_{H^1(\Omega(t))} + \|u\|_{L^2(\Gamma(t))} \|v\|_{L^2(\Gamma(t))} \\ &\leq C_D^2 \|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \|v\|_{H^{\frac{1}{2}}(\Gamma(t))} + \|u\|_{L^2(\Gamma(t))} \|v\|_{L^2(\Gamma(t))} \\ &\leq (C_D^2 + 1) \|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \|v\|_{H^{\frac{1}{2}}(\Gamma(t))}. \end{aligned}$$

For coercivity,

$$\begin{aligned} a(t; u, u) &= \int_{\Omega(t)} |\nabla(\mathbb{D}(t)u)|^2 + \|u\|_{L^2(\Gamma(t))}^2 && \text{(again with } \mathbb{E} = \mathbb{D}) \\ &= \|\nabla w\|_{L^2(\Omega(t))}^2 + \|u\|_{L^2(\Gamma(t))}^2 \\ &\geq C_1 \|w\|_{H^1(\Omega(t))}^2 && \text{(using (6.18))} \\ &\geq C_2 \|u\|_{H^{\frac{1}{2}}(\Gamma(t))}^2 \end{aligned}$$

by the trace theorem. Therefore, we have a unique solution $u \in W(H^{\frac{1}{2}}, H^{-\frac{1}{2}})$ to (6.22), and with $w(t) := \mathbb{D}(t)u(t)$ and the uniform bound (6.21), we find $w \in L^2_{H^1}$ (where $H^1 = \{H^1(\Omega(t))\}_{t \in [0, T]}$) satisfies the original PDE (6.13).

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