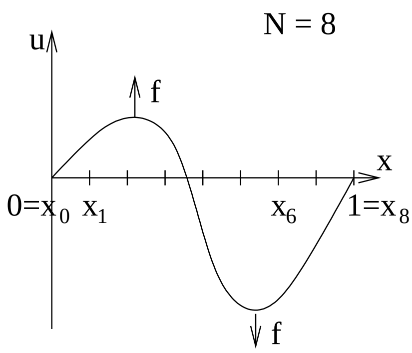
Dr. Björn Stinner Term 1, 2008/2009

MA398 Matrix Analysis and Algorithms
Exercises on Chapter 5

Exercise 5.1 (A):
Consider a rubber band attached in $(0, 0)$ and $(1, 0)$ (see picture). In every point $x \in [0, 1]$ a forces $f(x)$ (e.g. gravitation) acts on the band. Simple models yield that the displacement $u(x)$ in equilibrium is subject to the following differential equation:



$$(*) \quad -u''(x) = f(x).$$

For $N \in \mathbb{N}$ define a grid on $[0, 1]$ via the nodes $x_i = \frac{i}{N}$, $0 \leq i \leq N$, and let $h = \frac{1}{N}$. The goal is to approximate the solution u to $(*)$ on the grid, i.e. to compute the values $u_i := u(x_i)$.
For this purpose, $u''(x)$ is approximated with the following differential quotient:

$$D_h^2 u(x) := \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

As a discretisation of $(*)$, derive the following system of linear equations:

$$-D_h^2 u(x_i) = f(x_i), \quad 1 \leq i \leq N-1.$$

Insert the known boundary values $u_0 = u(x_0)$ and $u_N = u(x_N)$ so that an $(N-1) \times (N-1)$ system for the unknown u_i, $1 \leq i \leq N-1$ is obtained.

Exercise 5.2 (B):
Using LU (without pivoting) compute the growth factor of

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

Remark: applying pivoting would not change the rows anyway...

Exercise 5.3 (B):
A matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n \times n}$ is called **strictly diagonal dominant** if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for all } i \in \{1, \dots, n\}.$$

Show that such a matrix is invertible and its LU factorisation exists.
For this purpose, show that the remaining matrix $U_{k+1}^{(k)} = (u_{ij}^{(k)})_{i,j=k+1}^n$ after step k of the Gaussian elimination without pivoting still is strictly diagonal dominant.

Exercise 5.4 (B):
Assume that the LU factorisation of a regular matrix $A \in \mathbb{C}^{n\times n}$ is known and suppose that also the matrix

$$B := \begin{pmatrix} A & u \\ v^T & \gamma \end{pmatrix} \in \mathbb{C}^{(n+1)\times(n+1)} \quad \text{with } \gamma \in \mathbb{C},\ u, v, \in \mathbb{C}^n$$

is regular.
Using the LU factorisation of A, develop an algorithm to compute the LU factorisation of B. The computational cost should be of the order $O(n^2)$ and, hence, less than the cost for the Gaussian elimination of B (which is $\Theta(n^3)$).

Exercise 5.5 (B):
Find the eigenvalues, corresponding eigenspaces, and the determinant of a Householder reflection matrix

$$Q = I - 2vv^* \in \mathbb{C}^{n\times n}, \qquad \text{where } \|v\|_2 = 1.$$

Hint: consider $V := \text{span}\{v\}$ and its orthogonal complement.

Exercise 5.6 (A):
Consider again the triangular matrix in Ex. 5.2. Recall that the matrices L, U from the LU factorisation without pivoting preserve the structure in the sense that they are triangular in addition to their usual properties.
Show that the QR algorithm with Householder reflections does not preserve this structure but leads to a 'fill-in' so that more off-diagonal elements become non-zero. For this purpose, it is sufficient for this purpose to consider the case $n = 3$ and to perform one step of QR.

Exercise 5.7 (C):
Consider the real orthogonal matrix $G = \left(\begin{smallmatrix} c & s \\ -s & c \end{smallmatrix}\right)$ with $c^2 + s^2 = 1$. Find $c, s \in \mathbb{R}$ such that $Q\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} r \\ 0 \end{smallmatrix}\right)$ with $r = \sqrt{x_1^2 + x_2^2}$.
Such 'Givens rotations' can be used instead of Householder reflections to compute a QR factorisation. Sketch how this technique can be employed to 'rotate away' sub-diagonal entries of a matrix.

GOOD LUCK!