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Simulation of cluster point processes without edge effects

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Abstract

Cluster point processes are important models for many biological data sets, and their simulation is often required for purposes of model exploration and statistical inference. The usual direct method of simulation requires the generation of the parent point process over a region larger than the actual observation window, since one has to allow for all possible parents giving rise to observed daughter points, and some of these parents may fall outwith the observation window. When there is no *a priori* bound on the distance between parent and child then one has to take care to control approximations arising from edge effects. In this paper we present a simulation method which requires simulation only of those parent points actually giving rise to observed daughter points, thus avoiding edge effect approximation. The idea is to replace the cluster distribution by one which is conditioned to plant at least one daughter point in the observation window, and to modify the parent process to have an inhomogeneous intensity exactly balancing the effect of the conditioning. We furthermore show how the method extends to cases involving infinitely many potential parents, for example gamma–Poisson processes and shot-noise G-Cox processes, allowing us to avoid approximation due to truncation of the parent process.

Keywords: Cox process, shot-noise process, gamma–Poisson process.

1 Introduction

Many applications give rise to point patterns exhibiting clustering. For example point patterns of positions of trees and plants often show clustering, either because plants tend to emerge where the soil has particular properties or because of seed setting mechanisms [5, 4, 12]. Such clustered point patterns have traditionally been modelled by Cox processes or cluster point processes. A classical example concerns the spatial distribution of insect larvae, typically exhibiting clustering around the egg masses from which the larvae are hatched [10]. Neyman and Scott [11] introduced the use of cluster processes in the study of clusters in locations of galaxies.

Cox processes [6] are Poisson processes with a random intensity and are natural models to consider when clustering is due to spatially varying features such as soil properties.

Cluster point processes are generated by first constructing a point process of unobserved (perhaps marked) parent points, each of which give rise to a random number of observed daughter points. The process of daughter points is the cluster process; it arises naturally as a model for point patterns where clustering is due to mechanisms such as seed setting. The construction is reminiscent of the Boolean model for a random set and so cluster point processes are sometimes viewed as germ-grain random set processes which happen to have grains which are random point patterns.

Sometimes the two classes of point processes coincide. This is the case for Neyman-Scott processes [11] (cluster processes in which the parent process is a homogeneous Poisson process and the daughters are dispersed independently around their parent according to some density function k) in the particular case when the number of daughter from each parent follows a Poisson distribution. Such a *Cox Poisson cluster process* can be viewed as a Cox process in which the random intensity λ is the sum of the intensity functions centred around the parent points:

$$\lambda(\cdot) \quad = \quad \sum k(\cdot - x) \tag{1}$$

where the sum is over all points x in the random pattern of parent points. A further generalization of Cox Poisson cluster processes is the class of shot-noise Cox processes [3, 4, 14]. For such processes the parent process is a general marked Poisson process, and the clusters are themselves Poisson processes with intensities $k(\cdot - x)$ allowed to depend on the mark m of the parent point x. Again such processes can be viewed as Cox processes.

Note that marking the cluster parents allows us to consider parent Poisson processes producing locally dense infinite realizations of marked parent points, so long as the final intensity of daughter points is almost surely bounded on bounded regions (this amounts to a requirement that most of the marks m lead to very low intensities of daughter points).

Having chosen a cluster process model to model some biological phenomenon, there is commonly a requirement to simulate realisations from this process (this arises for purposes of inference such as envelope generation or simulated maximum likelihood estimation, or to enable one to study the types of point patterns arising from a particular model). The conventional direct means of simulation requires generation of parent points over a region larger than the observation window, since parents outside the observation window can give rise to daughters within. If the daughter density k has unbounded support then any such realisation will be approximate, because of the edge effects arising from neglecting parents outside the simulation region. Moreover, when simulating shot-noise Cox processes yielding infinite numbers of parent points in bounded regions, it is not only necessary to simulate the parent process on a larger region but also to truncate the parent process to a finite number of points. It can be awkward to make quantitative assessment of the effects of truncation or edge effects.

In this paper we present a method for simulating Cox Poisson cluster point processes or shot-noise Cox processes which avoids incurring edge or truncation effects. The method is based on thinning and exploits the observation that the point processes under study are both cluster point processes and Cox processes. In a nutshell, we simulate only those parents giving rise to daughters in the observation window. In this way the simulation requirement is reduced so as to concern only finitely many parents.

In order to provide a clear understanding of the method we start by developing the method in the simple Neyman-Scott case with independent marks. After presenting the theory for the method in a general framework, we then show how the method can be applied so as to avoid numerical integration of the offspring dispersion intensity k. Finally we show how to use the method to conduct exact simulation from shot-noise Cox processes with infinitely many possible parents in bounded regions. Examples include gamma–Poisson processes [14] and shot-noise G-Cox processes [3]. After the development of each case of the method we present the algorithm in pseudo–code and we conclude the paper by presenting three illustrative examples and a discussion of the results.

2 Simulation procedure

In this section we describe how to avoid edge-effects when simulating a cluster point process (or more generally a germ-grain process) within a bounded observation window. We do this by simulating only those points of the parent process which actually contribute daughters lying within the observation window, and we further modify the simulation by using rejection sampling to modify the family-size distribution accordingly. We begin by formulating the method in its simplest form, which is the easiest to understand but which requires calculations of various integrals which are potentially burdensome. In Section 2.2 we then describe the theoretical framework, and in Subsection 2.3 we use this together with rejection sampling to show how to avoid calculations of burdensome integrals. In Subsection 2.4 we show how to apply the method to the case in which the parent point pattern is no longer locally finite. This last generalisation enables us to avoid both edge-effects and truncation errors when simulating e.g. Gamma-Poisson [14] and shot-noise G Cox processes [3, 4], where the parent processes give rise to only finitely many points in bounded subsets albeit producing infinitely many effects (mostly very small) at shot-noise level.

2.1 The method in its simplest form

Let Ψ be a homogeneous marked Poisson process on $\mathbb{R}^2$ with positive intensity λ and with points marked independently by non-negative marks of distribution M. Thus Ψ can be viewed as a Poisson process on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $\lambda \times \text{area} \times M$. Following stochastic geometry notation [13] we write a typical marked point of Ψ as $[x; m]$.

We require that the mark distribution has a finite mean:

$$\mathbb{E}[m] \quad = \quad \int m M(\mathrm{d}m) \quad < \quad \infty\,.$$

Consider a kernel K from $\mathbb{R}^2$ to $\mathbb{R}^2$, subject to the further finiteness condition $\int_{\mathbb{R}^2} K(x, A)\,\mathrm{d}x < \infty$ for bounded subsets $A \subset \mathbb{R}^2$, and use this to define a random measure ν based on the marked point process Ψ by

$$\nu(A) \quad = \quad \sum_{[x;m]\in\Psi} mK(x, A)\,. \tag{2}$$

Note that the finiteness conditions are equivalent to the following requirement on the random measure ν:

$$\mathbb{E}[\nu(A)] = \mathbb{E}\left[\sum_{[x;m]\in\Psi} mK(x, A)\right] \quad < \quad \infty\,. \tag{3}$$

The kind of process in which we are primarily interested is a Cox process Φ with random intensity measure ν, and we suppose our requirement is to simulate only the restriction of Φ to a fixed bounded window $W \subset \mathbb{R}^2$. We do this by representing the Cox process Φ as a cluster process (in fact, a Neyman-Scott process) based on Ψ. In the following we will use the following standard terminology: (marked) points $[x; m]$ of Ψ are called *parent points*, points y of Φ are called *daughter points*, and for convenience we will assume that the kernel K has a density k with respect to the Lebesgue measure, i.e. $K(x, A) = \int_A k(x, u)\,\mathrm{d}u$. One can think of Φ as a germ-grain process with the germs of Ψ being the parent points, and the grains being the locally finite sets of daughters.

Every parent point in Ψ has the potential to contribute daughters to Φ within W, but Φ is finite so only finitely many parents actually do contribute. The number of daughters lying within W and contributed by a given parent $[x; m] \in \Psi$ follows a Poisson distribution with mean $mK(x, W)$ (conditional on the marked parent point $[x; m]$), and the daughters are distributed independently over K with density proportional to $k(x, \cdot)$. Condition (3) is equivalent to the requirement that the accumulated point pattern of daughters of all parents is locally finite. Integrating out the mark m, we arrive at the conditional probability $p(x)$ that a parent point located at x contributes daughters to Φ within the window W:

$$p(x) \quad = \quad 1 - \int_{\mathbb{R}_+} \exp(-mK(x,W))\,\mathrm{d}M(m) \quad = \quad 1 - \mathfrak{L}_M(K(x,W))\,, \quad (4)$$

where $\mathfrak{L}_M$ is the Laplace transform of the mark distribution M.

The marks are independent, so the restriction of the parent point process Ψ to Ψ' (those unmarked parent points which end up contributing daughters to Φ) can be considered as produced by independent position-dependent thinning of the parent points lying in Ψ with retention probability $p(\cdot)$. Independent thinning of a Poisson process yields a Poisson process; therefore Ψ' is an inhomogeneous Poisson process with intensity function

$$f_1(x) \quad = \quad \lambda p(x) \quad = \quad \lambda[1 - \mathfrak{L}_M(K(x,W))]\,, \qquad x \in \mathbb{R}^2\,. \qquad (5)$$

The total number of parent points of Ψ contributing daughters to Φ within W is thus Poisson distributed with mean $\int_{\mathbb{R}^2} f_1(x)\,\mathrm{d}x$, and the daughters from a contributing parent $x' \in \Psi'$ form a *conditioned* randomized Poisson process on W such that the unconditioned variant has intensity $m'k(x',\cdot)\mathbb{I}_W(\cdot)$, where the mark m' is randomized with distribution M. The condition to be applied is that the process is non-null within W (the trivial realization $\Phi = \emptyset$ arises when no parent points of Ψ contribute daughters within W).

We can now state the most direct version of the simulation procedure, supposing we have already been given a function `InhomogeneousPoisson`(f_1, W) which returns a set of points forming a realization of a Poisson process on W with intensity function $f_1(\cdot)$:

```
def SimpleSim(λ,k,W,M):
    define f_1 : x ↦ λ[1 − L_M(K(x,W))]
    Ψ = InhomogeneousPoisson(f_1,W)
    Φ = ∅
    for x in Ψ:
        Φ = Φ ∪ SimDaughters(k,W,x,M)
    return Φ
```

where the function `SimDaughters` is given by:

```
def SimDaughters(k,W,x,M):
    ρ = K(x,W)
    n = 0
    while n is nonzero:
        draw m using M
        draw n using Poisson(mρ)
    draw y_1,...,y_n using k(x,·) restricted to W
    return {y_1,...,y_n}
```

Here we have presented the simplest possible version of `SimDaughters`, based on naïve rejection sampling. Notice that `InhomogeneousPoisson` has to be implemented either by employing a general (and possibly expensive) rejection-sampling approach or by using transformation methods based on special properties of f_1.

2.2 The theoretical framework

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Euclidean spaces (more general formulations are possible, but we omit these). We think of $\mathcal{X}$ as the set of possible locations for parents (including their marks if relevant), and $\mathcal{Y}$ as the set of possible locations for daughters. Let μ be a nonnegative measure on $\mathcal{X}$: we use this as an intensity measure to produce a Poisson process Ψ of parent points. Let $K(x,\cdot)$ be a probability kernel from $\mathcal{X}$ to $\mathcal{Y}$, so that $K(x,A)$ is a probability measure for each fixed parent $x \in \mathcal{X}$, and is a measurable function of x for each measurable subset $A \subseteq \mathcal{X}$. We shall use $K(x,A)$ to produce a Poisson point sub-pattern of daughters in $\mathcal{Y}$ for each parent x.

Thus we obtain a cluster process Φ in $\mathcal{Y}$ as the union of all the Poisson sub-patterns produced as x ranges over the Poisson pattern of parents. As is well-known, such a cluster process is also a Cox process with random intensity measure

$$\sum_{x\in\Psi} K(x,\cdot)$$

obtained by summing over the parent points.

We are interested in simulating that part of this cluster process Φ which lies in a fixed measurable window $W \subseteq \mathcal{Y}$. A thinning argument shows that the sub-pattern of parents each with at least one daughter in W is also Poisson but with intensity measure

$$(1-\exp(-K(x,W)))\,\mu(\,\mathrm{d}x)\,.$$

Each of these retained parents x possesses a family of daughters lying in W: the pattern of locations of daughters is given by a Poisson process of intensity measure $K(x,\cdot)$ restricted to W, but conditioned to have at least one point. Consequently the pattern of locations of daughters of x can be simulated by sampling the total number N from a Poisson distribution of mean $K(x,W)$ conditioned to be positive (this can be implemented simply as a rejection algorithm, or in more sophisticated ways); then distributing each of the N daughters independently and identically over W using the probability distribution which is the renormalization of $K(x,\cdot)$ restricted to W.

This is the simplest version of the method. The variant discussed in the previous Subsection 2.1 is adapted to allow for marks for the parent points. In abstract terms we write the parent space as $\mathcal{X}\times\mathbb{R}_+$, the intensity measure of Ψ as $\nu\otimes M$ where M is the marginal probability distribution for the marks, and suppose the kernel is of the form $K([x;m],\cdot) = mK(x,\cdot)$. Discarding the marks, the parent locations contributing daughters form a Poisson process on $\mathcal{X}$ with intensity measure

$$1-\int_{\mathbb{R}_+}\exp(-mK(x,W))\,\mathrm{d}M(m)$$

as described in Eq.(5) above. Each contributing parent x then contributes daughters to W according to a conditioned randomized Poisson process as described in Subsection 2.1 above, with unconditioned random intensity measure

$$mK(x, \cdot \cap W)$$

where m is random with distribution M.

Now we consider how rejection techniques might be used to finesse away calculations of integrals (such as those used to renormalize each of the $K(x, \cdot)$ restricted to W). Suppose in the unmarked case that $\tilde{K}(x, \cdot)$ is a nonnegative kernel dominating $K(x, \cdot)$, by which we mean that we can write

$$K(x, A) \quad = \quad \int_A \rho(x, y) \tilde{K}(x, \mathrm{d}y) \tag{6}$$

for some measurable $\rho \leq 1$. Suppose that $\tilde{K}$ satisfies the unmarked version of condition (3). It may be possible to choose the dominating kernel $\tilde{K}(\cdot)$, and a dominating window $\tilde{W}_x$ (possibly depending on the parent x), such that it is much easier to carry out the above construction for $\tilde{K}(\cdot)$ and $\tilde{W}_x$.

Theorem 2.1 *Consider the point pattern $\tilde{\Phi}$ of daughters $\tilde{y}$ produced from parents $x \in \Psi$ by applying the above procedure using dominating kernel $\tilde{K}(x, \cdot)$ and dominating window $\tilde{W}_x$. We shall refer to this as the* dominating process. *Suppose each daughter point $\tilde{y} \in \tilde{\Psi}$ is thinned with retention probability $\rho(x, \tilde{y})\mathbb{I}_W(\tilde{y})$, where x is the parent point; this thinning to be an independent thinning when considered as applied to the point process of daughters marked by their parents. The resulting pattern of retained daughters, has the distribution of a realization of an application of the procedure using kernel $K(x, \cdot)$ and fixed window W.*

Proof: The most immediate proof uses simple coupling ideas. To each of the daughters $\tilde{y} \in \tilde{\Phi}$ in the dominating process, assign an independent mark $U_{\tilde{y}}$ uniformly distributed over $[0, 1]$. Implement an independent thinning by retaining $\tilde{y}$ daughter of x whenever $\rho(x, \tilde{y}) < U_{\tilde{y}}$. The resulting cluster process produces daughter patterns using kernel $K(x, \cdot)$ by virtue of Eq.(6). Each daughter pattern is a realization of an inhomogeneous Poisson process, since this property is preserved under independent thinning. If parents are thinned by retaining only those parents which contribute daughters to W then we obtain the construction giving rise to the direct method in Subsection 2.1 above. On the other hand exactly the same result is obtained by using the thinning criterion $\rho(x, \tilde{y}) < U_{\tilde{y}}\mathbb{I}_{W_x}(y)$ and then intersecting the result with the original window W, and this is as specified in the theorem. $\square$

We describe the algorithmic implications in the next subsection. Because we have formulated this for differing parent and daughter spaces $\mathcal{X}$ and $\mathcal{Y}$, we can apply the method to the case in which parent points are marked, even including cases such as shot-noise Cox processes represented as Neyman-Scott processes with locally *infinite* parent patterns, for which however only finitely many parents contribute daughters to any bounded window.

Corollary 2.1 *Consider the case of marked parent points; we write the parent space as $\mathcal{X}\times\mathbb{R}_+$, the intensity measure of Ψ as $\nu\otimes M$ where M is now some non-negative measure, and suppose the kernel is of the form $K([x;m],\cdot) = mK(x,\cdot)$, still satisfying condition (3). Let $\tilde{K}([x;m],\cdot) = m\tilde{K}(x,\cdot)$ be a dominating kernel also satisfying condition (3), so that*

$$K(x,A) \quad = \quad \int_A \rho(x,y)\tilde{K}(x,\, dy)$$

as in Eq.(6) above, and let $\tilde{W}_x$ be a dominating window with $W \subseteq \tilde{W}_x$. Then the thinning construction of Theorem 2.1 may be applied: consider the point pattern $\tilde{\Phi}$ of daughters produced using the dominating kernel $m\tilde{K}(x,\cdot)$. Suppose daughter $y \in \tilde{\Phi}$ of parent x is thinned with retention probability $\rho(x,\tilde{y})\mathbb{I}_W(\tilde{y})$. Then the retained daughters form a point pattern which has the distribution which would be produced by using the original kernel $mK(x,\cdot)$.

The strength of this corollary lies in the fact that we need take no account of the mark m in the thinning procedure. We discuss the algorithmic implications in Subsection 2.4.

2.3 The modified method

The method of Subsection 2.1 is simple and appealing, but it involves calculation of integrals of k (evaluation of $K(x,W)$ in the definition of $f_1(x)$) which will either require special properties of f_1 or use potentially expensive rejection sampling. We now describe how the simulation procedure can be modified using the framework of Subsection 2.2 so that calculation of the integrals can be avoided or at least simplified, at the expense of using rejection sampling at a lower level of the algorithm (hopefully in a efficient form, depending on precise details of the process). Working out the method of Subsection 2.2 in a 2-dimensional Euclidean context, consider a larger window $\tilde{W} \subseteq \mathbb{R}^2$ with $W \subseteq \tilde{W}$ and a dominating kernel $\tilde{K}$ with density $\tilde{k}$, i.e. $k(x,u) \leq \tilde{k}(x,u)$ for all $u \in \tilde{W}$ and $x \in \mathbb{R}^2$ (for simplicity we suppose here that $\tilde{W}$ does not depend on the parent in question). The window $\tilde{W}$ and kernel $\tilde{K}$ should be chosen so that $\tilde{K}(x,\tilde{W})$ can be evaluated cheaply for all $x \in \mathbb{R}^2$ and so that $\int_{\mathbb{R}^2} f_2(x)\,\mathrm{d}x$ can be evaluated, where

$$f_2(x) \quad = \quad \lambda[1 - \mathfrak{L}_M(\tilde{K}(x,\tilde{W}))]\,, \qquad x \in \mathbb{R}^2\,. \tag{7}$$

Moreover we need to be able to draw from the probability distribution given by normalization of f_2. In the examples given below, $\tilde{W}$ is chosen to be a disc containing W and $\tilde{k}$ to be a constant on $\tilde{W}$, whereby we get a usable expression for $\tilde{K}(x,\tilde{W})$.

The modified method consists of:

- replacing W and K by $\tilde{W}$ and $\tilde{K}$ in the method described in Section 2.1, i.e. first simulate a cluster process $\tilde{\Phi}$ with daughter intensity $\tilde{k}$ on $\tilde{W}$,

- then thinning $\tilde{\Phi}$ to produce a cluster process $\Phi_{\tilde{W}}$ with daughter intensity k on $\tilde{W}$
- and finally restricting $\Phi_{\tilde{W}}$ to W in order to get a realisation of Φ on W.

Theorem 2.1 assures us that the result has the correct distribution. The important implementation point is that $k(x, u)$ is involved only in evaluation of the ratios $k(x, u)/\tilde{k}(x, u)$; this offers us the chance of avoiding potentially expensive integration of k.

In order to present the modified method in pseudo-code we assume a function `Thin(`Φ, p`)`, which independently thins the points of the process Φ with retention probability $p(\cdot)$, where $p : \mathbb{R}^2 \to [0, 1]$. (Of course `Thin(`$\Phi, p$`)` itself is easy to code.) The modified procedure can then be described as follows.

```
def ModifiedSim(λ, k̃, k, W̃, W, M):
    define f₂(x) ↦ λ[1 − 𝔏_M(K̃(x, W̃))]
    Ψ̃ = InhomogeneousPoisson(f₂,W̃)
    Φ = ∅
    for x̃ in Ψ̃:
        Φ₀ = SimDaughters(k̃,W̃,x̃,M) ∩ W
        Φ = Φ ∪ Thin(Φ₀,k(x̃,·)/k̃(x̃,·))
    return Φ
```

Here `SimDaughters` is implemented as described in Subsection 2.1.

2.4 The general method

The marked point process Ψ used to construct the process Φ in the previous sections can be viewed as a random measure on $\mathbb{R}^2$, placing an atom at each point $x \in \Psi$ with corresponding atom mass given by the respective mark m. Passing to an infinitely divisible discrete random measure, the random intensity measure ν for the Cox process Φ can be generalised to

$$\nu(A) \quad = \quad \int_{\mathbb{R}^2} K(x, A)\, \mathrm{d}\zeta(x) \quad = \quad \sum_{m\delta_x \text{is an atom of } \zeta} mK(x, A)\,, \tag{8}$$

where $\zeta = \sum m\delta_x$ is a discrete random measure such that the parameters (x, m) of its atoms form an inhomogeneous Poisson process on $\mathbb{R}^2 \times \mathbb{R}_+$. Since ζ is a discrete random measure we get a representation similar to that of Equation (2), but with the difference that ζ can have infinitely many atoms in bounded areas. The finiteness condition (3) can be rewritten as

$$\mathbb{E}\left[\sum_{m\delta_x \text{is an atom of } \zeta} mK(x, A)\right] \quad = \quad \mathbb{E}\left[\int_{\mathbb{R}^2} K(x, A)\, \mathrm{d}\zeta(x)\right] \quad < \quad \infty\,, \tag{9}$$

where the $\mathrm{d}\zeta$ term takes the weight of each atom into consideration, to ensure convergence of the sum over countably many atoms mostly of diminishingly small weight.

Consider e.g. the case where ζ is a gamma measure, i.e. a completely random measure where $\zeta(A)$ follow a gamma distribution for every bounded A [7]. This is an example of a measure that is purely atomic, with an infinite number of atoms in any bounded set, but only finitely many atoms of mass greater than a fixed $\epsilon > 0$. Wolpert and Ickstadt [14] have used Cox processes with intensity measures of the form (8) and ζ a gamma measure to model positions of trees. Other examples of general random measures ζ arise from the shot-noise G measure as defined in [3] and used for weed modelling in [4]. We follow [3] and use the term *shot-noise Cox process* for a Cox process with intensity measure of the form (8).

Both [14] and [3] describe how to simulate shot-noise Cox processes using the techniques developed in [2]. Just as with the simple cluster process model it is necessary to simulate ζ on a larger region than W, but furthermore it is only possible to simulate a finite number of atoms in any bounded area (see [3] for a further discussion of this). Using a generalisation of the methods described in the previous section it is, however, possible to simulate shot-noise Cox processes without edge effects and without having to truncate ζ. Generalizing the simple case, the atoms $m\delta_x$ contributing daughters to W can be represented as an inhomogeneous Poisson process of points x, m on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity

$$f_3(x,m) \quad = \quad \lambda(x,m)[1 - \exp(-mK(x,W))]\,. \tag{10}$$

(In effect we have absorbed the mark into the three-dimensional locations of the points.) Note that the intensity of the locations (x, m) is now allowed to be inhomogeneous. The expected total number of daughter points, contributed by all of the infinitely many potential parent points taken together, is given by $\int_{\mathbb{R}^2}\int_{\mathbb{R}_+} f_3(x,m)\,\mathrm{d}m\,\mathrm{d}x$, and this is finite as a consequence of the finiteness condition (9). Furthermore the daughters contributed to W by a point (x, m) of this inhomogeneous Poisson process themselves form a Poisson process with intensity $mk(x,\cdot)$ but then thinned in accordance with a conditioning that they have each to place a positive number of points in W, following the procedure indicated in Corollary 2.1. It is simple to modify `ModifiedSim` accordingly: one has to simulate marks as well as locations as the result of `InhomogeneousPoisson` and to pass a constant mark to `SimDaughters` rather than a mark distribution. In effect we are generating the intensity measure for the Cox process Φ using a Poisson process of parent points stretching over a three-dimensional space $\mathbb{R}^2 \times \mathbb{R}_+$, and such that marks for the resulting parent points are determined as functions of the point locations rather than being random.

```
def GeneralSim(λ, k̃, W̃, W):
    define f̃₃(x,m) ↦ λ(x,m)[1 − exp(−mK̃(x,W̃))]
    Ψ̃ = InhomogeneousPoisson(f̃₃,W̃ ∪ ℝ₊)
    Φ = ∅
    for (x̃,m̃) in Ψ̃:
        Φ₀ = SimDaughters(k̃,W̃,x̃,δ_m̃) ∩W
        Φ = Φ ∪Thin(Φ₀, k(x̃,·)/k̃(x̃,·))
    return Φ
```

Notice that we dispense with the integration previously implicit in the use of the Laplace transform in the functional argument to InhomogeneousPoisson; this reflects the fact that the marks are now deterministic components of the point locations in $\mathbb{R}^2 \times \mathbb{R}_+$.

3 Examples

Here are three brief descriptions of simulation examples using these ideas.

3.1 Deterministic marks

Our first example concerns simulation of a simple Neyman-Scott process on the square $W = [-0.5, 0.5]^2$. Specifically the process Ψ of parent points is a homogeneous Poisson process with positive intensity λ, the marks are non-stochastic and of common value 1, and the dispersion intensity for the daughter process is Gaussian, i.e.

$$k(x,u) \quad = \quad \frac{\gamma}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\|u-x\|^2\right) . \tag{11}$$

We apply the modified method specified in Section 2.3.

Construct $\tilde{W} = \{x \in \mathbb{R}^2 : \|x\| \leq 1/\sqrt{2}\}$ as the smallest disc that covers W, and choose for the dominating kernel density

$$\begin{aligned} \tilde{k}(x,u) &= \mathbb{I}_{\tilde{W}}(u) \times \max_{u\in\tilde{W}} k(x,u) \\ &= \mathbb{I}_{\tilde{W}}(u) \times \tfrac{\gamma}{2\pi\sigma^2} \exp\left(-\mathbb{I}_{\tilde{W}^c}(x) \times \tfrac{1}{2\sigma^2}\left(\|x\| - \tfrac{1}{\sqrt{2}}\right)^2\right) , \end{aligned}$$

Figure 1 graphs the kernel density and the dominating kernel density for $x = 1.05$, $\gamma = 10$ and $\sigma = 0.25$.

Since the marks are all identically equal to 1, their Laplace transform is $\mathfrak{L}_M(s) = \exp(-s)$, and a simple integration of $\tilde{k}(x,u)$ with respect to u shows that the intensity of parents contributing daughters in $\tilde{W}$ is given by

$$f_2(x) \quad = \quad \lambda\left[1 - \exp\left(-\frac{\gamma}{4\sigma^2} \exp\left(-\mathbb{I}_{\tilde{W}^c}(x) \times \frac{1}{2\sigma^2}\left(\|x\| - \frac{1}{\sqrt{2}}\right)^2\right)\right)\right] . \tag{12}$$

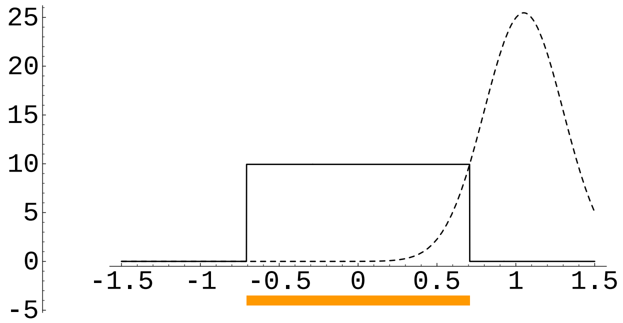

Figure 1: Profiles of the dispersion kernel density $k((1.05, 0), (\cdot, 0))$ (dotted) and the dominating density $\tilde{k}((1.05, 0), (\cdot, 0))$ (solid). $\tilde{W}$ is indicated by the thick line from $-1/\sqrt{2}$ to $1/\sqrt{2}$. The density $\tilde{k}((1.05, 0), (\cdot, 0))$ only needs to dominate $k((1.05, 0), (\cdot, 0))$ in the region $\tilde{W}$.

The function f_2 is well-behaved (radially symmetric, decays fast to zero), allowing the efficient use of numerical integration in `InhomogeneousPoisson` to find the intensity of parent points that contribute daughters to $\tilde{W}$. Figure 2 shows a realisation of the Neyman-Scott process with $\lambda = 10$, $\gamma = 10$ and $\sigma = 0.25$, with each daughter point linked to the corresponding parent point. Figure 3 shows the realization of the daughters alone without the parents.

3.2 Gamma distributed marks

Suppose that in the previous example we replace the constant marks by independent and identically distributed marks. Then we can use the same $\tilde{k}$ and $\tilde{W}$. Assume that the marks are independent and gamma distributed with shape parameter ω and scale parameter β, then the Laplace transform of the marks is $\mathcal{L}_M(s) = (1 + \beta s)^{-\omega}$, and f_2 becomes

$$f_2(x) = \lambda \left[1 - \left(1 + \frac{\beta\gamma}{4\sigma^2} \exp\left(-\mathbb{I}_{\tilde{W}^c}(x) \times \frac{1}{2\sigma^2} \left(\|x\| - \frac{1}{\sqrt{2}} \right)^2 \right) \right)^{-\omega} \right]. \tag{13}$$

Figure 4 shows a realisation of the resulting point process using parameters of the previous example and mark parameters $\beta = 0.25$, $\omega = 4$ together with links of daughters to their parent points. (Parameters are chosen to maintain the same overall intensity of points between examples, though random variation is markedly increased.) Figure 5 shows the realization of the daughters alone without the parents.

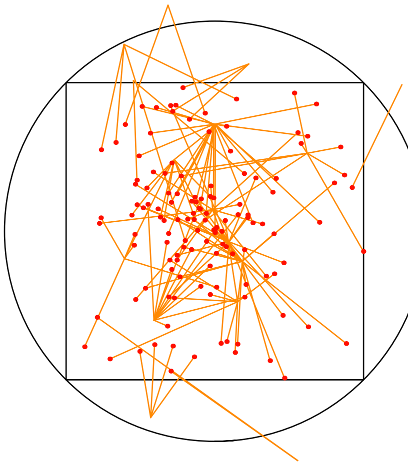

Figure 2: Realisation of Neyman-Scott process (dots); parent points which contribute daughters are presented as linked to their offspring.

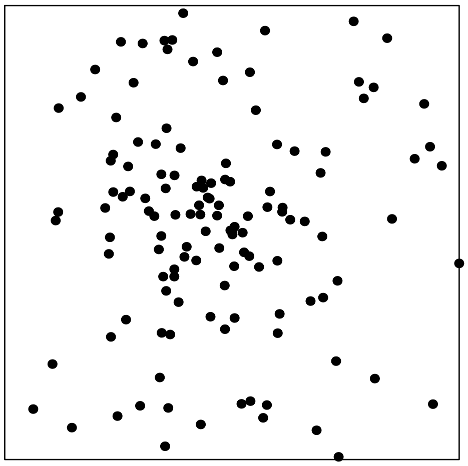

Figure 3: Realisation of Neyman-Scott process without links to parents.

3.3 Gamma-Poisson processes

As an example of how to simulate a general shot-noise Cox process without edge effects and truncation error, consider a gamma-Poisson process as defined in [14] and [3]. The parent process is a Poisson process on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity

$$\lambda(x, m) = \kappa m^{-1} \exp(-m/\beta), \qquad \kappa, \beta > 0 \tag{14}$$

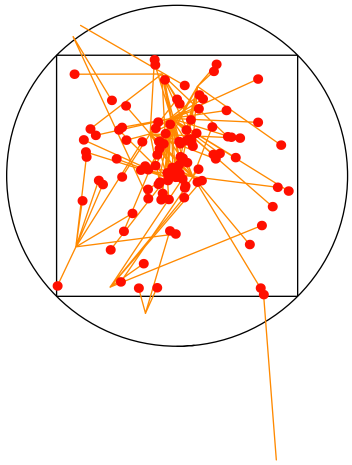

Figure 4: Realisation of case with gamma-distributed marks (dots); parent points which contribute daughters are presented as linked to their offspring.

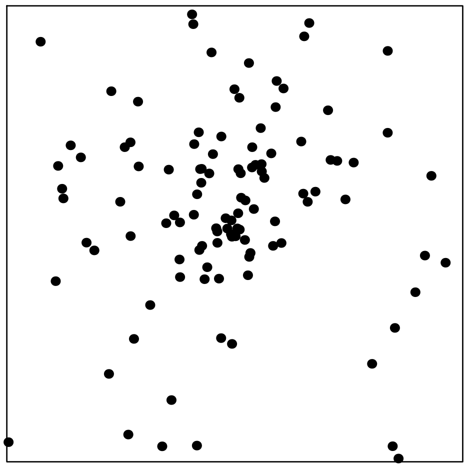

Figure 5: Realisation of case with gamma-distributed marks without links to parents.

If we choose k and W as in the first example, we can use the same $\tilde{k}$ and $\tilde{W}$, and the intensity of the point pattern of parents contributing points to $\tilde{W}$ is

$$f_3(x, m) \quad = \quad \lambda(x, m)\left[1 - \exp(-m\tilde{K}(x, \tilde{W}))\right] \qquad (15)$$

While the original intensity $\lambda(x, m)$ has infinite integral over $\mathbb{R}^2 \times \mathbb{R}_+$, the singularity at $m = 0$ is removed on multiplication by $1 - \exp(-m\tilde{K}(x, \tilde{W}))$ (which also vanishes at $m = 0$). This allows us to use a thinning method to simulate the inhomogeneous Poisson process of parents conditioned to contribute to the dominating disk under the dominating kernel.

Figure 6 shows a realisation of the resulting point process using parameters of the previous example and mark parameters $\beta = 0.25$, $\kappa = 40$ together with links of daughters to their parent points. Figure 7 shows *all* potential parent points (as determined by whether or not they contribute to the dominating disk under the dominating kernel), with attached segments indicating the size of their marks. Figure 8 shows the realization of the daughters alone without the parents.

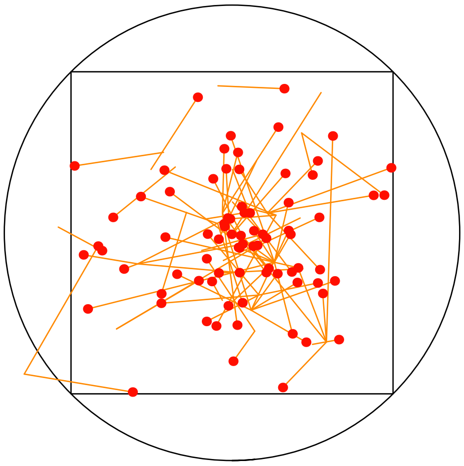

Figure 6: Realisation of gamma-Poisson process (dots); parent points which contribute daughters are presented as linked to their offspring.

4 Discussion

We remark that there is a direct generalization of the above to cover the case of a general Neyman-Scott cluster process with family size distribution F, probability generating function $G(s)$. In that case the pattern of parents contributing daughters to the observation window W is inhomogeneous Poisson with intensity function

$$f_4(x) \quad = \quad 1 - G(H(x, W))\,,$$

where $H(x, \cdot)$ is the dispersion probability kernel for the daughters (in the unmarked Poisson case, $K(x, \cdot) = G'(1)H(x, \cdot)$). For each contributing parent a

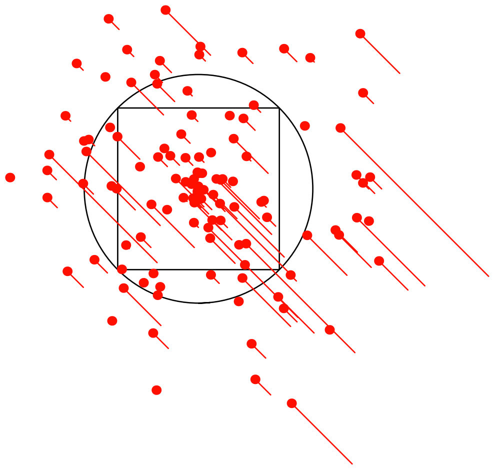

Figure 7: Potential parents, with attached segments indicating size of marks.

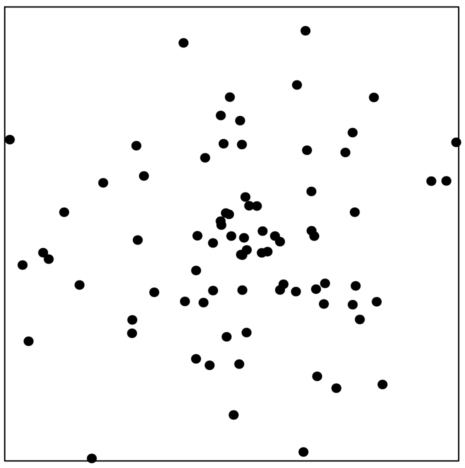

Figure 8: Realisation of gamma-Poisson process (dots) without links to parents.

daughter pattern has to be simulated:

```
def SimNonPoissonDaughters(H,W,x,F):
    repeat:
        draw N using F
        draw {y_1,...,y_N} using H(x,·)
        until {y_1,...,y_N} ∩ W is non−empty
    return {y_1,...,y_N} ∩ W
```

This is of course substantially more clumsy as we lose invariance under independent thinning.

Wolpert and Ickstadt [14] demonstrate the use of Markov chain Monte Carlo to estimate posterior statistics, such as the intensity function governing observed daughter locations. The ideas discussed here allow us to adapt their method to provide a treatment which avoids truncation and edge-effects: we will discuss this in further work.

It is of course the case that in practical circumstances one can avoid the use of the methods discussed here simply by taking a large enough guard region (and, in the case of Subsection 3.3, by sampling parents with marks exceeding a small enough threshold): careful choice of this will control the resulting errors. However the methods described here allow us to give a treatment exact in at least this respect, without undue effort or complexity.

Finally we note the resemblance of the methods described here to those of "perfect simulation in space" for lattice and spatial interaction models as found in [1, 8, 9]. Both situations can be viewed as potentially infinite simulation tasks, which are rendered feasible because all but a finite amount of work can be shown irrelevant to what one requires to observe. However the problem for interaction models is rendered more challenging because interactions propagate from one unobserved point to another: the cluster processes considered here are much simpler to deal with.

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