Valuing Bermudan Options When Asset Returns are Lévy Processes

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Abstract

Evidence from the financial markets suggests that empirical returns distributions, both historical and implied, do not arise from diffusion processes. A growing literature models the returns process as a Lévy process, finding a number of explicit formulae for the values of some derivatives in special cases.

Practical use of these models has been hindered by a relative paucity of numerical methods that can be used when explicit solutions are not present. In particular, the valuation of Bermudan options in problematical.

This paper investigates a lattice method that can be used when the returns process is Lévy, based upon an approximation to the transition density function of the Lévy process. We find alternative derivations of the lattice, stemming from alternative representations of the Lévy process, that may be useful if the transition density function is unknown or intractable.

We apply the lattice to models based on the variance-gamma and NIG processes. We find that the lattice is able to price Bermudan style options to acceptable accuracy.

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1 Introduction

In the standard derivative pricing framework the prices of derivatives are given as the expected discounted value of their terminal payoff, where the discounting is with respect to some numeraire and the expectation is taken under an equivalent martingale measure associated with the numeraire. The evolution of a set of state variables that together determine the payoff to the option and the value of the numeraire is specified by writing down the stochastic processes they follow under the equivalent martingale measure.

When the underlying state variables are Lévy processes explicit solutions for derivative prices are often unavailable. Numerical methods include Monte Carlo, Fourier transform (FFT) methods (Carr and Madan (99) [18]), and PDE approaches (the 'method of lines' of Albanese, Jaimungal and Rubisov (01) [2], [3], and a PIDE approach of Hirsa and Madan (01) [32]). These approaches have some limitations. Unless a Lévy process is amenable to simulation Monte Carlo methods may not be used. In any case American and Bermudan style options are hard to value by Monte Carlo, despite recent progress. American options may also give problems for FFT methods. PDE methods may be used with the variance-gamma model (Madan, Carr and Chang (98) [37]) but their use with other processes has not been established.

We want to be able to value Bermudan options. Lattice methods have been used extensively to compute prices in models with a small number of state variables following Itō diffusion processes.[1] We introduce a lattice method for pricing with Lévy processes, with branching probabilities found from the transition density function. We apply the Lévy lattice to value Bermudan options in models driven by variance-gamma (VG) and normal inverse Gaussian (NIG) processes.

The lattice is consructed from Lévy density. We relate it to subordinated Brownian motion, the generating triple and to the time copula of the Lévy process. We demonstrate that in certain situations the Lévy lattice can be an effective method for valuing European and Bermudan instruments in models driven by a one-dimensional Lévy process. *En passant* we derive some results on the time copulas of subordinated Brownian motions.

In the next section we briefly review some facts about Lévy processes, and recall the VG and NIG processes. In section three we discuss numerical procedures. We describe the Lévy lattice giving alternative derivations of the lattice branching probabilities. Section four presents numerical results in pricing European and Bermudan options and section five concludes.

2 Lévy Processes

Three main types of Lévy processes have been investigated in the finance literature to model asset return's processess; the variance-gamma (VG) process, the

[1] Das (99) [23] describes a lattice method for a process incorporating a discrete jump variable.

normal inverse Gaussian (NIG) process, and the generalised hyperbolic (GH) process. The NIG process is a special case of the GH process. It is the only subclass closed under convolutions. The VG process can be characterised as a limit of GH processes.

The full generalised hyperbolic model has been described and investigated by, for instance, Eberlein (00) [25], Eberlein and Keller (95) [26], Rydberg (99) [47], and Bibby and Sørensen (01) [14]. It has been mainly used to model historical stock returns. Its use as a valuation tool is limited by its relative intractability, although several of the authors mentioned above extract option prices in models based on these processes. In this paper we focus on the more tractable VG and NIG preocesses.

A number of authors have specifically investigated derivative pricing, rather than the modelling of stock returns *per se*. Amongst these are Chan (99) [20], Benhamou (00) [13], Gallucio (01) [27], Lewis (01) [35], Carr and Wu (01) [19] and Albanese, Campolieti, Carr and Lipton (00) [1]. However, most work focuses on situations where explicit solutions may be obtained, rather than on numerical solutions.

We review some facts about Lévy processes in general and then the VG and NIG processes in particular.

The generating triple of a Lévy process The Lévy-Khintchine representation of a (one dimensional) Lévy process $L = (L_t)_{t\geq 0}$ determines the generating triple (a, ν, γ) that defines the characteristic function of L_1,

$$\mathbb{E}\left[\mathrm{e}^{izL_1}\right] = \exp\left(-\frac{1}{2}az^2 + i\gamma z + \int_{\mathbb{R}} \left(\mathrm{e}^{izx} - 1 - izx1_D(x)\right)\nu(\mathrm{d}x)\right), \quad (1)$$

where $a, \gamma \in \mathbb{R}$, $\alpha > 0$, $D = \{x \mid |x| \leq 1\}$, and ν is a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \left(\left|x^2\right| \wedge 1\right)\nu(\mathrm{d}x) < \infty$. ν is called the Lévy measure. $\nu(B)$ is the expected number of jumps over a unit time period whose sizes are contained in the set B. If $\nu \equiv 0$, a and γ are the volatility and drift of a Brownian motion. If ν is not trivial, different choices for the normalising function $1_D(x)$ lead to different values of γ.

If ν has finite mass then it is the Lévy measure of a compound Poisson process with intensity $\lambda = \int_{\mathbb{R}} \nu(\mathrm{d}x)$. A Lévy process L can be approximated as a sum of a Brownian motion and a compound Poisson process. Truncated Lévy processes, where jumps of size less than ε have been removed, for some $\varepsilon > 0$, are significantly easier to handle numerically. They have been studied by, for instance, Matacz (00) [40] and Boyarchenko and Levendorskii (00) [16].

Lévy processes and subordinated Brownian motion Lévy processes are semimartingales. A semimartingale $X = (X_t)_{t\geq 0}$ can be represented as a time-changed Brownian motion

$$X_t = w_{h_t} \quad (2)$$

where w is a Brownian motion and h is a stochastic time change (Monroe (78) [42]). Below we consider only the case when the time-change h is an increasing Lévy process, a subordinator. Then X will also be a Lévy process.

Hurst, Platen and Rachev (97) [33] give a review of modelling stock returns as subordinated Brownian motions. Recent studies include Madan (01) [36], Ané and Geman (00) [7] and Geman, Madan and Yor (01a) [28], (01b) [29], (02) [30].

The Variance-Gamma Process The VG model of asset returns, due to Madan and Seneta (90) [39], has been developed and extended in a number of papers including Madan and Milne (91) [38], Madan, Carr and Chang (98) [37] and Carr, Geman, Madan and Yor (03) [17]. It has formulae for option prices in terms of special functions.

The VG process $L^{\mathrm{VG}} = \left(L_t^{\mathrm{VG}}\right)_{t\geq 0}$ has three parameters: $\mu \in \mathbb{R}$, $\sigma > 0$, $\upsilon > 0$. It is pure jump with Lévy density

$$\nu^{\mathrm{VG}}(\mathrm{d}x) = \kappa^{\mathrm{VG}}(x)\,\mathrm{d}x = \frac{\exp\left(\mu x/\sigma^2\right)}{\upsilon\,|x|}\exp\left(-\frac{1}{\sigma}\sqrt{\frac{2}{\upsilon}+\frac{\mu^2}{\sigma^2}}\,|x|\right)\mathrm{d}x \tag{3}$$

and characteristic function

$$\mathbb{E}\left[\exp\left(iuL_1^{\mathrm{VG}}\right)\right] = \left(\frac{1}{1-i\mu\upsilon u+\frac{1}{2}\sigma^2\upsilon u^2}\right)^{\frac{t}{\upsilon}}. \tag{4}$$

The density function $f_t^{\mathrm{VG}}(x)$ of L_t^{VG} is

$$f_t^{\mathrm{VG}}(x;\mu,\sigma,\upsilon) = \frac{2\exp\left(\mu x/\sigma^2\right)}{\upsilon^{\frac{t}{\upsilon}}\sqrt{2\pi}\sigma\Gamma\left(\frac{t}{\upsilon}\right)}\left(\frac{x^2}{2\sigma^2/\upsilon+\mu^2}\right)^{\frac{t}{2\upsilon}-\frac{1}{4}} K_{\frac{t}{\upsilon}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2\left(2\sigma^2/\upsilon+\mu^2\right)}\right) \tag{5}$$

where $K_\upsilon(z)$ is the modified Bessel function of the second kind.

A VG process L^{VG} can be represented as a Brownian motion with drift μ and variance σ^2 subordinated to a gamma process h with parameters $(t, \upsilon t)$. The density $f_t^h(x)$ of h_t conditional on $h_0 = 0$ is

$$f_t^h(x) = \frac{x^{\frac{t}{\upsilon}-1}\exp\left(-\frac{x}{\upsilon}\right)}{\upsilon^{\frac{t}{\upsilon}}\Gamma\left(\frac{t}{\upsilon}\right)}. \tag{6}$$

The Normal Inverse Gaussian Process The NIG model has been discussed and developed by Eberlein and Keller (95) [26], Barndorff-Nielsen (95) [9], Barndorff-Nielsen (97a) [10], Barndorff-Nielsen (97b) [11] and Rydberg (97) [46], amongst others.

An NIG process $L^{\mathrm{NIG}} = \left(L_t^{\mathrm{NIG}}\right)_{t\geq 0}$ has four parameters: $\mu \in \mathbb{R}$, $\delta \in \mathbb{R}_+$ and $0 \leq |\beta| < \alpha$. It is convenient to define $\gamma = \sqrt{\alpha^2-\beta^2}$. The density function of

L_t^{NIG} is

$$f_t^{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu)=\frac{\delta t\alpha}{\pi}\mathrm{e}^{\delta t\gamma}\frac{K_1\left(\alpha\sqrt{\delta^2t^2+(x-\mu t)^2}\right)}{\sqrt{\delta^2t^2+(x-\mu t)^2}}\mathrm{e}^{\beta(x-\mu t)} \tag{7}$$

where K_1 is the modified Bessel function of the second kind with index 1. The class of normal inverse Gaussian distributions is closed under convolution.

The characteristic triplet of the NIG distribution is $\left(a^{\mathrm{NIG}},0,\nu^{\mathrm{NIG}}\right)$ where the Lévy measure ν^{NIG} has density

$$\nu^{\mathrm{NIG}}(\mathrm{d}x)=k^{\mathrm{NIG}}(x)\,\mathrm{d}x=\frac{\delta\alpha}{\pi|x|}\mathrm{e}^{\beta x}K_1(\alpha|x|)\,\mathrm{d}x \tag{8}$$

and

$$a^{\mathrm{NIG}}=\mu+2\pi^{-1}\delta\alpha\int_0^1\sinh(\beta x)K_1(\alpha x)\,\mathrm{d}x.$$

The NIG process can be represented as a Brownian motion subordinated to an inverse Gaussian (IG) process h. The density function of h_t is

$$f_t^h(x;\alpha,\beta,\delta)=\frac{\delta t}{\sqrt{2\pi}}x^{-3/2}\exp\left(-\frac{1}{2x}(\gamma x-\delta t)^2\right),\qquad x>0$$

If $h\sim\mathrm{IG}(\alpha,\beta,\delta t)$ and $\varepsilon\sim\mathrm{N}(0,1)$, then $x=\mu t+\beta h+\sqrt{h}\varepsilon$ has an NIG distribution with density $f_t^{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu)$.

2.1 Time Copulas and Lévy Processes

In this section we briefly describe the copula and show how a set of copulas arises from stochastic process. Standard references include Nelson (99) [43] and Joe (97) [34]. There are several instances of copula methods being used for valuation purposes. Cherubini and Luciano (00) [21] have shown how in principle, given a particular set of barrier options, an implied copula for a terminal time can be determined from market data. Copulas have used for modelling correlated default in multi-asset credit risk. Other financial applications are discussed in Bouyé *et al.* (00) [15].

The copula is a joint distribution function for standard uniform variates U_i, $i=1,\ldots,N$,

$$C(u_1,\ldots,u_N)=\Pr(U_1\leq u_1,\ldots,U_N\leq u_N),\ U_i\sim U[0,1],\ i=1,\ldots,N. \tag{9}$$

Any continuous distribution function[2] $F(x_1,\ldots,x_N)$ of random variables $X_1,\ldots,X_N$ can be decomposed into a composition of the individual marginal distribution functions $F_i(x_i)$ and a copula function $C(u_1,\ldots,u_N)$,

$$F(x_1,\ldots,x_N)=C(F_1(x_1),\ldots,F_N(x_N)). \tag{10}$$

[2] With minor technical rstrictions.

This decomposition is essentially unique. All possible dependency information is captured by the copula.

We need the following property of copulas in the sequel: If $Y_i = f_i(X_i)$, $i = 1, 2$, for monotonic increasing functions f_i, then the copula of Y_1 and Y_2 is the same as the copula of X_1 and X_2.

Given a stochastic process X with $X_0 = 0$, one may consider the joint distributions $F_{t_1,t_2}(x_1, x_2)$,

$$F_{t_1,t_2}(x_1, x_2) = \Pr[X_{t_1} < x_1, X_{t_2} < x_2], \tag{11}$$

of the process at times t_1 and t_2, $0 < t_1 < t_2$. The family of copulas $C_{t_1,t_2}(u_1, u_2)$, implicitly defined as

$$F_{t_1,t_2}(x_1, x_2) = C_{t_1,t_2}(F_{t_1}(x_1), F_{t_2}(x_2)), \tag{12}$$

where $F_t(x)$ is the (conditional) marginal distribution function of X_t, is called the time copula for the process X. The concept of the time copula was introduced by Darsow, Nguyen and Olsen (92) [22] who wrote down the time copula for a standard Brownian motion. A set of copulas $C_{t_1,t_2}(u_1, u_2)$ defines a time copula if and only if

$$C_{t_1,t_3}(u_1, u_2) = \int_0^1 \partial_2 C_{t_1,t_2}(u_1, u)\, \partial_1 C_{t_2,t_3}(u, u_2)\, \mathrm{d}u \tag{13}$$

for all times $t_1 \leq t_2 \leq t_3$, where ∂_i denotes differentiation with respect to the ith variable.

In this section we find an expression for the time-copula of a subordinated stochastic process and demonstrate how the Wiener and Ornstein-Ulhenbeck time copulas appear as special cases. We start by investigating diffusion processes and then turn to general 1-factor Lévy processes.

2.1.1 The time copula of a 1-factor diffusion

A martingale diffusion process can always be represented as a time changed standard Brownian motion. We use this property to investigate its time copula. Suppose that $M = (M_t)_{t \geq 0}$ is a (local) martingale diffusion process with $M_0 = 0$ and $\langle M, M \rangle_\infty = \infty$. Set $h_t = \langle M, M \rangle_t$. Then (Revuz and Yor (99) [44]) we have $M_t = z_{h_t}$ for a standard Brownian motion z. h is an increasing deterministic right-continuous function of t, with $h_0 = 0$. It is a (deterministic) time change.

Conversely, diffusion processes X whose solutions can be expressed in the form $X_t = z_{h_t}$, where h_t is a deterministic time change, are local martingales.

We find the time copula of $X = z_{h_t}$ in terms of the function h. Write Φ for the standard normal distribution function. For each t the random variable

$$W_t = \frac{1}{\sqrt{h_t}} z_{h_t} \tag{14}$$

is standard normal, hence the process U where $U_t = \Phi(W_t) = \Phi\left(\frac{1}{\sqrt{h_t}} z_{h_t}\right)$ has uniform marginals. Since the map $x \mapsto \Phi\left(\frac{1}{\sqrt{h_t}} x\right)$ is monotonic increasing,

the time copula for U is the same as the time copula for X. We can readily identify this copula. Consider an increment $\Delta X_{s,t} = z_{h_t} - z_{h_s}$, where h is locally deterministic. Since $\Delta X_{s,t} \sim \mathrm{N}(0, h_t - h_s)$ is normally distributed with mean zero and variance $h_t - h_s$, we have

$$X_t = \sqrt{h_t} W_t = \sqrt{h_s} W_s + \sqrt{h_t - h_s}\varepsilon_{s,t} \tag{15}$$

where $\varepsilon_{s,t} \sim \mathrm{N}(0,1)$ is standard normal and independent of W_s. Then for some $u_s = \Phi(w_s)$

$$\begin{aligned}
\Pr[U_t < v \mid U_s = u_s] &= \Pr\left[w_t < \Phi^{-1}(v) \mid W_s = w_s\right] && (16)\\
&= \Pr\left[\varepsilon_{s,t} < \frac{\sqrt{h_t}}{\sqrt{h_t - h_s}}\Phi^{-1}(v) - \frac{\sqrt{h_s}}{\sqrt{h_t - h_s}} w_s\right] && (17)\\
&= \Phi\left(\frac{\sqrt{h_t}}{\sqrt{h_t - h_s}}\Phi^{-1}(v) - \frac{\sqrt{h_s}}{\sqrt{h_t - h_s}}\Phi^{-1}(u_s)\right). && (18)
\end{aligned}$$

Hence the time-copula $C^X_{s,t}(u,v)$ of X is given by

$$\begin{aligned}
C^X_{s,t}(u,v) &= \Pr[U_s < u, U_t < v] = \int_0^u \Pr[U_t < v \mid U_s = u_s]\,\mathrm{d}u_s && (19)\\
&= \int_0^u \Phi\left(\frac{\sqrt{h_t}}{\sqrt{h_t - h_s}}\Phi^{-1}(v) - \frac{\sqrt{h_s}}{\sqrt{h(t) - h_s}}\Phi^{-1}(u_s)\right)\mathrm{d}u_s. && (20)
\end{aligned}$$

We call the process U where $U_t = \Phi\left(\frac{1}{\sqrt{h_t}} z_{h_t}\right)$ a uniform h time-changed standard Brownian motion. Each time-change h determines a dependency structure. Conversely every diffusion dependency structure determines an h. This case covers stochastic volatility and other non-jump specifications.

For a standard Brownian motion z, $h_t = t$, so the time copula is

$$C^{\mathrm{z}}_{s,t}(u,v) = \int_0^u \Phi\left(\frac{\sqrt{t}}{\sqrt{t-s}}\Phi^{-1}(v) - \frac{\sqrt{s}}{\sqrt{t-s}}\Phi^{-1}(q)\right)\mathrm{d}q, \tag{21}$$

as in Darsow, Nguyen and Olsen (92) [22]. The copula (21) is called the Gaussian time copula.

We can also find the time copula when X is an Ornstein-Uhlenbeck process. Let $\mathrm{d}X_t = \alpha(\mu - X_t)\,\mathrm{d}t + \sigma \mathrm{d}z_t$ with $z_0 = 0$. This SDE has a solution

$$X_t = a(t) + b(t)\,\tilde{z}_{h_t} \tag{22}$$

where, in particular, $h_t = \mathrm{e}^{2\alpha t} - 1$. The time copula of X is the same as the time copula of $(\tilde{z}_{h_t})_{t \geq 0}$ and is given by (20) with $h_t = \mathrm{e}^{2\alpha t} - 1$. This special case of (20) coincides with the result of Bouyé *et al.* (00) [15].

Conversely, since X_t is normally distributed, $X_t \sim \mathrm{N}\left(a(t), b^2(t)\,h_t\right)$, given an $\mathrm{e}^{2\alpha t} - 1$ time-changed uniform standard Brownian motion U we may define an Ornstein-Uhlenbeck process X as

$$\frac{X_t - a(t)}{b(t)\sqrt{h(t)}} = \Phi^{-1}(U_t). \tag{23}$$

2.1.2 The time copula of a Lévy process

We investigate the time copula of a Lévy process L through the subordinator h of its subordinator representation. Since a stock price S_t is an increasing function of a returns process $L_t = w_{h_t}$, the time copula of S is the same as the time copula of L. Furthermore, since for each fixed t the Brownian motion w is a monotonic increasing function[3] of a standard Brownian motion z, we need only investigate processes where $L_t = z_{h_t}$ are representable as subordinated standard Brownian motions.

Suppose h_t has a distribution function $H_t(\alpha) = \Pr[h_t < \alpha]$ and joint distribution function $H_{s,t}(\alpha, \beta) = \Pr[h_s < \alpha, h_t < \beta]$ for times s and t. We derive an expression for the time copula of L. The marginal distributions F_t^L of L_t is

$$F_t^L(l) = \Pr[L_t < l] = \int_0^\infty \Pr[z_\alpha < l] H_t(\mathrm{d}\alpha) \tag{24}$$

$$= \int_0^\infty \Phi\left(\frac{l}{\sqrt{\alpha}}\right) H_t(\mathrm{d}\alpha). \tag{25}$$

Let $F_t^{\mathrm{z}}(l) = \Pr[z_t < l]$ and $F_{s,t}^{\mathrm{z}}(l,k) = \Pr[z_s < l, z_t < k]$ be the marginal and the joint distribution functions of a standard Brownain motion z. The joint distribution functions $F_{s,t}^L$ are

$$F_{s,t}^L(l,k) = \Pr[L_s < l, L_t < k] = \int_0^\infty \int_0^\infty \Pr[z_\alpha < l, z_\beta < k] H_{s,t}(\mathrm{d}\alpha, \mathrm{d}\beta) \tag{26}$$

$$= \int_0^\infty \int_0^\infty F_{\alpha,\beta}^{\mathrm{z}}(l,k) H_{s,t}(\mathrm{d}\alpha, \mathrm{d}\beta) \tag{27}$$

$$= \int_0^\infty \int_0^\infty C_{\alpha,\beta}^{\mathrm{z}}\left(F_\alpha^{\mathrm{z}}(l), F_\beta^{\mathrm{z}}(k)\right) H_{s,t}(\mathrm{d}\alpha, \mathrm{d}\beta) \tag{28}$$

$$= \int_0^\infty \int_0^\infty C_{\alpha,\beta}^{\mathrm{z}}\left(\Phi\left(\frac{l}{\sqrt{\alpha}}\right), \Phi\left(\frac{k}{\sqrt{\beta}}\right)\right) H_{s,t}(\mathrm{d}\alpha, \mathrm{d}\beta). \tag{29}$$

Hence the time copula $C_{s,t}^L(u,v)$ of L_t is

$$C_{s,t}^L(u,v) = F_{s,t}^L\left(\left(F_s^L\right)^{-1}(u), \left(F_t^L\right)^{-1}(v)\right) \tag{30}$$

$$= \int_0^\infty \int_0^\infty C_{\alpha,\beta}^{\mathrm{z}}\left(\widetilde{u}_{\alpha,s}, \widetilde{v}_{\beta,t}\right) H_{s,t}(\mathrm{d}\alpha, \mathrm{d}\beta), \tag{31}$$

where $\widetilde{u}_{\alpha,s}$ and $\widetilde{v}_{\beta,t}$ satisfy the relationship

$$u = \int_0^\infty \Phi\left(\sqrt{\frac{\alpha}{\gamma}}\Phi^{-1}\left(\widetilde{u}_{\alpha,s}\right)\right) H_s(\mathrm{d}\gamma), \tag{32}$$

$$v = \int_0^\infty \Phi\left(\sqrt{\frac{\beta}{\gamma}}\Phi^{-1}\left(\widetilde{v}_{\beta,t}\right)\right) H_t(\mathrm{d}\gamma). \tag{33}$$

[3] When the stock has positive expected returns.

The expression (31) is quite general, but unlikely to be useful except in special cases. For instance, if h is deterministic this reduces to $C^L_{s,t}(u,v) = C^{\mathrm{z}}_{h(s),h(t)}(u,v)$ as in the previous section.

3 Constructing the Lévy Lattice

Following Madan, Carr and Chang (98) [37], Barndorff-Nielsen and Shephard (00) [12], Eberlein and Keller (95) [26], *et cetera*, we model a stock price process S under the pricing measure as

$$S_t = S_0 \exp(rt + L_t - \omega t) \tag{34}$$

where L is Lévy process. The presence of the compensator ω, defined by $\mathrm{e}^{\omega} = \mathbb{E}[\exp(L_1)]$, ensures that $(S_t \mathrm{e}^{-rt})_{t\geq 0}$ is a martingale. For a VG process the compensator is $\omega = -\frac{1}{\nu}\ln\left(1-\mu\nu-\frac{1}{2}\sigma^2\nu\right)$. For a NIG process the compensator is $\omega = \mu + \delta\left(\gamma - \sqrt{\alpha^2-(\beta+1)^2}\right)$.

We wish to value derivatives on S. Monte Carlo methods may be available to value European path dependent options, especially by simulation via the subordinator representation. However it is much easier to value Bermudan or American options on a lattice. An efficient lattice method may also be significantly faster than a Monte Carlo method for valuing some types of path dependent options, such as barrier options. We focus here on developing lattice methods for the underlying Lévy process. From (34) a lattice for L will generate a lattice for S. The chief difficulty is to construct branching probabilities in the lattice. We do so from the transition density function of the Lévy process.

We also consider several alternative derivations for the brnching probabilities. These are from:

1. The Lévy triple of L
2. The representation of L as a subordinated Brownian motion
3. The time-copula representation of L.

Each of these routes has complications, discussed below, but may be preferable to the primary density construction in some circumstances.

Amin (93) [5] describes a lattice method for jump-diffusion processes. He applied it to the Merton (76) [41] model and to a model with a small number of discrete jump-sizes. His method obtains branching probabilities from the underlying transition density function. Although his lattice is restricted to compound Poisson processes, and works best for relatively small jump intensities, it is closely related to the Lévy lattice described in this paper. Schmidt (97) [48] discusses lattice methods for processes of the form $(f(z_{h_t}))_{t\geq 0}$, for particular functional forms for f, when the subordinator is deterministic.

In the first part of this section we describe the Lévy lattice. We describe alterntive derivations. The section ends with a discusson of implications for implied pricing methods.

3.1 The Lévy Lattice

When h_t is a stochastic subordinator the direct use of a Schmidt type lattice does not work. Instead we construct a multinomial lattice that converges to a Lévy process. The method generalises the Amin (93) lattice for compound Poisson processes.

We construct a discrete process $\widetilde{L} = \left(\widetilde{L}_t\right)_{t\geq 0}$, approximating L, directly from the transition density functions of L_t when these are known. We show that $\widetilde{L}$ can also be constructed from the Lévy-Khintchine representation of a Lévy process, a subordinator representation, or from a time copula. These alternative constructions may be appropriate in certain situations, discussed below.

The lattice is in discrete time $t \in \{0 = t_0 < t_1 < \ldots < t_M = T\}$, with $t_{i+1} - t_i = \Delta t$, a constant. Let $J = \{-D^{\max}, \ldots, U^{\max}\} \subset \mathbb{Z}$ for truncation bounds $D^{\max}, U^{\max} > 0$. $\widetilde{L}_{t_i}$ takes values in a set $\widetilde{l}_{i,j}$, $j \in J$, with

1. $\widetilde{l}_{i,j+1} - \widetilde{l}_{i,j} = \Delta l$, a constant for all i, and j, $j+1 \in J$;
2. $\widetilde{l}_{i+1,j} = \widetilde{l}_{i,j}$ for all i and $j \in J$;
3. $\widetilde{l}_{t_0}$ has value $\widetilde{l}_{0,0} = 0$.

Let $Q = \{-D, \ldots, U\} \subset \mathbb{Z}$, for bounds D,$U > 0$, be an index set, and let $K = \{k_q\}_{q\in Q} \subset \mathbb{Z}$ be indexed by Q, with $k_q < k_{q'}$ if $q < q'$, and $k_0 = 0$. For standard branching we could define $k_q = q$, $q \in Q$, but we see below that this branching specification has drawbacks. We discuss the choice of K shortly. Branching for $\widetilde{L}_t$ is stationary and additive. Branching from $\widetilde{l}_{i,j}$ is to the set $\left\{\widetilde{l}_{i+1,j+k} \mid k \in K,\ j+k \in J\right\}$. The order of branching is $|K| = D+U+1$. The branching probabilities $p_k = p_k(\Delta t, \Delta l) = \Pr\left[\widetilde{l}_{t_{i+1}} = \widetilde{l}_{i+1,j+k} \mid \widetilde{l}_{t_i} = \widetilde{l}_{i,j}\right]$, for $k \in K$ and $j+k \in J$, are independent of i and j (where defined).

Nodes (i,j) such that $j + U \leq U^{\max}$ and $j - D \geq -D^{\max}$ are said to be in the interior of the lattice. For nodes not in the interior of the lattice the branching is truncated. We assume, and ensure in practice, that the error in option prices due to this truncation is negligable. Branching probabilities $P = \{p_k\}_{k\in K}$ are determined so that in the interior of the lattice $\widetilde{L}_t$ converges to L_t as $(\Delta t, \Delta l) \to 0$.

A Brownian motion is α-stable with $\alpha = 2$. For these processes one may exploit the α-stable scaling property, setting $\Delta l = (\kappa \Delta t)^{\frac{1}{2}}$ for some constant κ, so that it is possible to define branching probabilities that do not depend on Δt or Δl. If L is α-stable we may define $\Delta l = (\Delta t)^{\frac{1}{\alpha}}$ but in general no such device is available and branching probabilities will depend on Δt.[4]

[4] Amin's compound Poisson lattice sets Δl to be a multiple of $\sqrt{\Delta t}$. This is an appropriate choice for modelling compound Poisson processes in which the jump intensity λ is relatively small. Amin notes that for fixed Δt the lattice is less accurate when λ is large.

Suppose that a derivative pays off at time $T = t_m$. If the payoff $H\left(L_{t_m}, t_m\right)$ depends solely on L_{t_m}, for instance if the stock price S_T depends solely on L_T and H depends only on S_T, then we may price the derivative on the lattice for $\widetilde{L}$.

The stock value at node (i, j) is $\widetilde{S}_{i,j}$, set to be

$$\widetilde{S}_{i,j} = \widetilde{S}_0 \exp\left(rt_i + \widetilde{l}_{i,j} - \widetilde{\omega} t_i\right) \tag{35}$$

where $\widetilde{\omega}$ is the discrete compensator on the lattice, $\mathrm{e}^{-r} \exp\left(\widetilde{\omega}\right) = \widetilde{W}$ where $\widetilde{W}$ is the value on the lattice at time 0 of a security paying off $\exp\left(\widetilde{L}_1\right)$ at time 1.

Write $\widetilde{c}_{i,j}$ for the value on the lattice of the derivative at time t_i when $\widetilde{L}_{t_i} = \widetilde{l}_{i,j}$. At time t_m set $\widetilde{c}_{m,j} = H\left(\widetilde{l}_{m,j}, t_m\right)$. In continuous time we have $c_t = \mathbb{E}_t\left[\mathrm{e}^{-r\Delta t} c_{t+\Delta t}\right]$ under the equivalent martingale measure with the accumulator account as numeraire.[5] On the lattice we use the discrete form

$$\widetilde{c}_{i,j} = \mathrm{e}^{-r\Delta t} \sum_{k \in K, j+k \in J} p_k \widetilde{c}_{i+1,j+k}. \tag{36}$$

Using (36) one iterates back through the lattice to obtain the option value $c_{t_0} = \widetilde{c}_{0,0}$ at time t_0. Values for American and Bermudan options can be computed by setting

$$\widetilde{c}_{i,j} = \max\left(\mathrm{e}^{-r\Delta t} \sum_{k \in K, j+k \in J} p_k \widetilde{c}_{i+1,j+k},\ H\left(\widetilde{l}_{i,j}, t_i\right)\right) \tag{37}$$

at each node where exercise, with payoff $H\left(\widetilde{l}_{i,j}, t_i\right)$, is possible.

Moment matching In the Black-Scholes world the value of a stock is a function of the values of a standard Brownian motion z,

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma z_t\right), \tag{38}$$

where $\frac{1}{2}\sigma^2$ is the compensator. Given a lattice for z, payoffs for options on S can be computed and discounted back. In a lattice for a standard Brownian motion z, setting $\Delta z = \sqrt{\kappa \Delta t}$ for some fixed κ, the set of branching probabilities P can be chosen independently of Δt and Δz. For branching of order three, P and κ can be chosen so that the first five moments of increments in $\widetilde{z}_t$ match those of increments in z_t.

Using results in Heston and Zhou (00) [31], Alford and Webber (01) [4] constructed a heptanomial lattice with branching of order seven that matched the first nine moments of w_t. Very fast convergence was obtained. Unfortunately

[5] Throughout, we assume that the interest rate, r, is constant.

simple moment matching methods cannot work for general Lévy processes. For instance, higher moments may simply not exist. Also, a lattice that matches the first N moments of a Lévy processes fails to distinguish between processes that differ in the $(N+1)$ th moment. Explicit solutions for higher order moments may not be known or may be difficult to compute. Even if moments of higher order can be found, finding probabilities to match them may be computationally complex.[6]

We deduce that it is not possible in general to use a set of branching probabilities $P = \{p_k\}_{k\in K}$ when $|K|$ is small, for instance, $|K| \leq 10$. We find it necessary in some case to use values of $D + U + 1$ of order several thousand. Once probabilities have been obtained, we nevertheless expected moments to be closely matched, where they are defined.

3.1.1 Construction from the density function

We obtain branching probabilities directly from the density function, when this exists. Obtaining probabilities from the density function implicitly attempts to fit to the characteristic function.

Let $f_{\Delta t}(l)$ be the density function of $\Delta L_t = L_{t+\Delta t} - L_t$. Given Q and K we choose $P = \left\{p_{k_q}\right\}_{q\in Q}$ with

$$p_{k_q} = f_{\Delta t}(k_q \Delta l)\, n_{k_q}, \tag{39}$$

where the normalising coefficients n_{k_q} are defined by

$$\begin{aligned} n_{k_q} &= \frac{1}{2}(k_{q+1} - k_{q-1})\,\Delta l,\ q \neq -D, U, \\ n_{k_U} &= \frac{1}{2}(k_U - k_{U-1})\,\Delta l, \\ n_{k_{-D}} &= \frac{1}{2}(k_{1-L} - k_{-L})\,\Delta l. \end{aligned} \tag{40}$$

The set P will not sum exactly to 1. We correct for this as follows. Let q_0 index the mode of $f_{\Delta t}(k_q \Delta l)$,

$$f_{\Delta t}(k_{q_0}\Delta l) = \max_{q\in Q} f_{\Delta t}(k_q \Delta l). \tag{41}$$

By setting $p_{k_{q_0}} = 1 - \sum_{q\in Q\setminus\{q_0\}} p_{k_q}$ we force the branching probabilities to sum to 1.[7]

[6] Chris Rogers has pointed out that moment matching is equivalent to fitting the characteristic function in a region of zero. Although appropriate for Brownian motion, moment matching will not fit the characteristic function of a general Lévy process away from zero.

[7] In practice the required correction is slight. Note that when f is smooth and Δl is sufficiently small we can arrange that $\sum_{q\in Q\setminus\{q_0\}} p_{k_q} < 1$ with only mild restrictions on $\{k_q\}$. With our choices for $\{k_q\}$ (see below), $p_{k_{q_0}}$ is large relative to the correction $1 - \sum_{q\in Q} p_{k_q}$.

3.1.2 Convergence

Suppose that $k_q = q$, $q \in Q$. When there are no truncation bounds and the order of branching is infinite, $J = Q = \mathbb{Z}$, the lattice variable $\widetilde{L}_{\Delta t}$ converges to $L_{\Delta t}$. For a measureable set $A \subset \mathbb{R}$ set $P(A) = \int_A \mathrm{d}F^{\mathrm{L}}_{\Delta t}$ and $\widehat{P}(A) = \sum_{k\Delta l \in A} p_k$. $\widehat{P}(A)$ is the probability mass given to branching from 0 to values in A over a time step Δt on the lattice. The error $P(A) - \widehat{P}(A)$ goes to zero as $\Delta l \to 0$.

When the order of branching is finite we define $D = D(\Delta l)$ and $U = U(\Delta l)$ as follows. For $\varepsilon > 0$ choose D and U so that $1 - F(U\Delta l) < \frac{1}{2}\varepsilon$ and $F(D\Delta l) < \frac{1}{2}\varepsilon$. Then $\widetilde{P}(A) = \sum_{k\Delta l \in A,\ k \in I} p_k$ is the new probability mass of A on the lattice. As $\Delta l \to 0$ the error $P(A) - \widehat{P}(A)$ is bounded by ε and goes to zero if $A \subset (-D\Delta l, U\Delta l)$.[8]

Similarly, when there are trunction bounds these can be set at levels to make the truncation error arbitrarily small.

3.1.3 Implementation

A simple Lévy lattice has $k_q = q$. When Δl is small enough to allow a fine enough mesh near zero, to simultaneously sample far enough into the tails requries D and U to become very large. To improve computation times the branching has to be modified.

For a given time step Δt choose Δl to give an exceptable error. Then define $k_q = k_q(p, k)$ as follows.[9] Given U and D set $n_U = \left\lfloor \frac{U}{p^k} \right\rfloor$ and $n_D = \left\lfloor \frac{D}{p^k} \right\rfloor$ for positive integers p, and k. For $q \leq n_U$ set $k_q = q$. For $q > n_U$ suppose $n_U j < q \leq n_U(j+1)$ for some $1 \leq j$. Then set

$$k_q = qp^j + \sum_{i=0}^{j-1} n_U p^i; \tag{42}$$

we define k_q for $q < 0$ similarly. This definition causes k_q close to zero to bunch up while allowing sampling into the tails of the density. Compared to computing over the entire set $\{-D, \ldots, U\}$ the computational saving out weights the additional error, in the sense that for a given level of computational effort, the error is reduced.

Defining branching probabilities via the density function has several drawbacks.

1. An arbitrary procedure has had to be used to force the probabilities to sum to 1, if required.

2. If the likelihood of large jumps is large, k_U and k_{-D} may have to be very large, making the lattice very wide as it evolves.

[8] The error also goes to zero when a modification to the probability at k_{q_0} is made.

[9] Other formulations are possible. If it is easy to compute percentiles of the distribution function $F_{\Delta t}(l)$ of ΔL_t we could chose k_q so that $F_{\Delta t}(k_q \Delta l) - F_{\Delta t}(k_{q-1}\Delta l)$ was approximately constant. D and U would be chosen so that $F_{\Delta t}(k_{-D}\Delta l)$ and $1 - F_{\Delta t}(k_U \Delta l)$ were sufficiently small. Our choice was determined by simplicity.

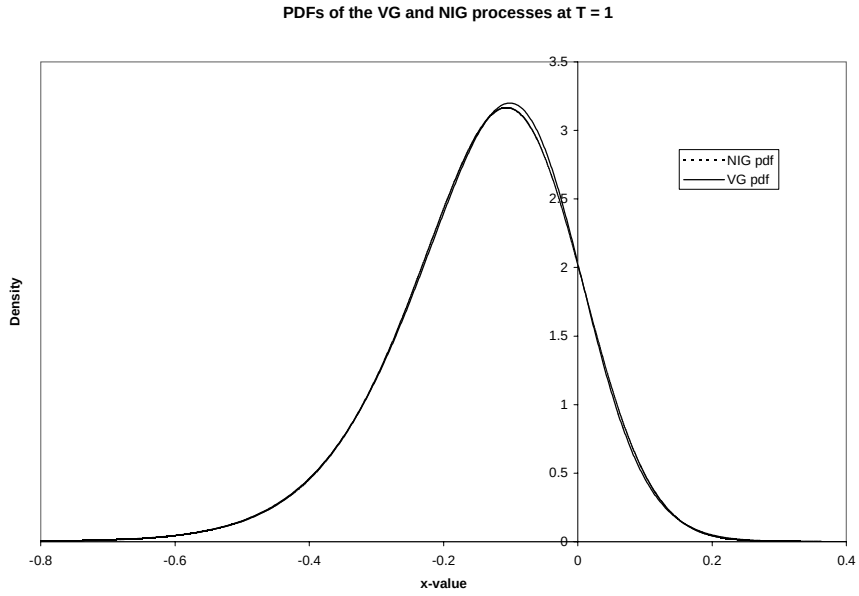

Figure 1: PDFs, VG and NIG processes, $\Delta T = 1$

3. If the likelihood of very small jumps is very large, unless Δl is very small the behaviour of ΔL_t near zero may not be captured.

4. Moments of ΔL_t are not automatically matched. Related to this, the compensator of the discrete process defined by P may not closely approximate the compensator of L.

The VG and NIG processes illustrate these problems. Plausible parameter values for the VG process, based on estimates from Madan, Carr and Chang, are $\mu = -0.14$, $\sigma = 0.12$ and $v = 0.2$. We took $r = 0.1$. For $\Delta t = 1$ the VG density function $f_1^{VG}(x; \mu, \sigma, v)$ is plotted in figure 1. The figure also shows the density of the NIG process at time 1 whose parameters $\{\alpha, \beta, \delta, \mu\}$ were chosen so that it has the same first four moments at time 1 as the VG process.[10] The resulting NIG density f_1^{NIG} closely resembles the VG density Both densities are skewed and the tails are fat.

With $\Delta l = 0.001$, setting $k_q = q$ and using $D = U = 4000$ we compute p_{k_q} by (39). The first four moments of the VG distribution are matched to 15dp (decimal places). The compensator is matched to 13dp. Analytical solutions for $c(X)$ are available in the VG model in terms of Bessel functions and hyper-

[10] To 5dp (decimal places) parameter values were $\alpha = 28.42141$, $\beta = -15.08623$, $\delta = 0.31694$, $\mu = 0.05851$. With these values the first two moments of the variance-gamma process were matched to 10dp, the third to 6dp and the fourth to 5dp.

X	Analytic solution	Numerical, $D = U = 4000$	Numerical, $D = 1000$, $U = 500$
90	19.09935	19.09936	19.09891
95	15.07047	15.07048	15.07001
100	11.37002	11.37002	11.36957
105	8.11978	8.11978	8.11937
110	5.42960	5.42960	5.42924
115	3.36543	3.36544	3.36515
120	1.92110	1.92110	1.92088

Table 1: European call prices: VG process, T=1, one time step

X	Reference value	Numerical $D = U = 4000$	Numerical $D = 1000$, $U = 500$
90	19.09330	19.09330	19.09367
95	15.06077	15.06077	15.06107
100	11.35994	11.35993	11.36014
105	8.11561	8.11561	8.11574
110	5.43723	5.43724	5.43729
115	3.38474	3.38475	3.38475
120	1.94359	1.94359	1.94359

Table 2: European call prices: NIG process, T=1, one time step

geometric functions (Madan, Carr and Chang (98) [37]). Although these have to be computed numerically they can be used to benchmark other numerical procedures. Results are shown in table 1. These European call option values agree to 5dp with the analytic formula.

Table 2 shows option values computed for the NIG process.[11] With $\Delta l = 0.001$ and $D = U = 4000$ the first four moments of the NIG process were matched to 14dp. The compensator was also correct to 14dp. These values are similar to the values obtained from the VG model.

Tables 1 and 2 also show options values when $D = 1000$ and $U = 500$, corresponding to truncating the distribution at -1 and 0.5. These values are accurate only to 3dp. The moments of the discrete distribution agree with the exact moments only to 3sf (significant figures). Values at the truncation levels are $f_1^{VG}(-1) \sim$ 1.8e-4 and $f_1^{VG}(0.5) \sim$ 1.3e-5; the truncation level is not far enough into the tails to give high accuracy.

These results indicate that direct numerical integration can value European options when T is sufficiently large. Difficulties are encountered if the time horizon is too small. Values of the density become arbitrarily large near zero as the time horizon decreases. Figure 2 shows the VG and NIG densities when $\Delta t = 0.1$. Although for fixed $x > 0$, $f_{\Delta t}(x \mid \Delta t) \to 0$ as $\Delta t \to 0$, the densities

[11] Reference values for the NIG process are computed by numerical integration of the payoff function against the NIG density using $\Delta l = 0.0001$ in the range $[-6, 6]$.

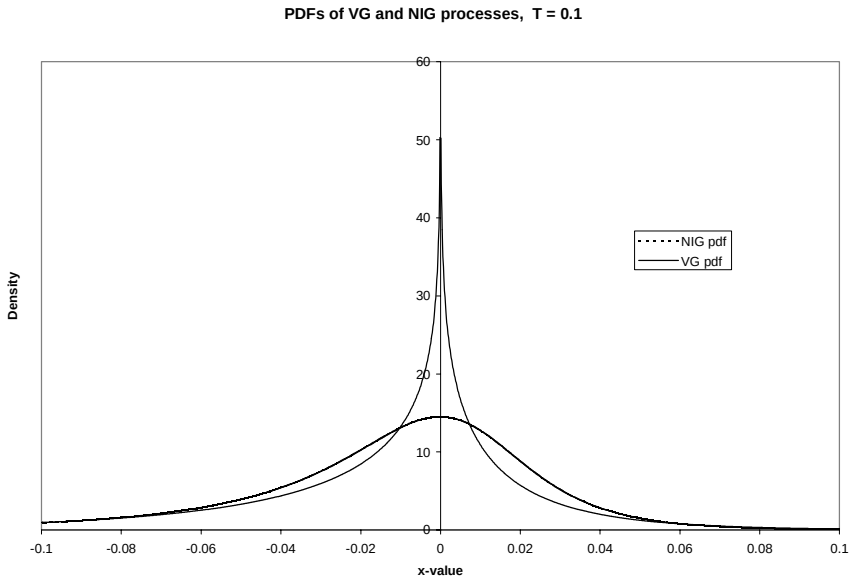

Figure 2: PDFs, VG and NIG processes, $\Delta t = 0.1$

do not go to zero quickly in Δt. Sharp peaks cause difficulties for numerical integration methods.

3.2 Alternative Derivations of the Lattice

The previous section described the construction of the Lévy lattice. If the density is known and tractable then this construction would usually be used. However, there are circumstances when alternative derivations of the branching probabilites may be preferable or necessary. We discuss these here.

3.2.1 Construction from the generating triple

By matching to the density function we are indirectly matching to the generating triple of the Lévy process, in particular, matching to the Lévy measure. We now try matching directly to the generating triple. This approach may be suitable if the Lévy process is compound Poisson. Monte Carlo methods may also work well for these processes (see Asmussen and Rosiński (01) [8] for a review), but it is much easier to value Bermudan or American options using a lattice method.

Consider a 1-dimensional Lévy process L with generating triple (a, ν, γ). We define a compound Poisson process L^{CP} approximating L whose generating

triple is $\left(a, \nu^{\mathrm{CP}}, \gamma\right)$. On a grid $\{k\Delta l\}_{k\in\mathbb{Z}}$ with fixed step $\Delta l > 0$ we define κ_k by

$$\kappa_k \Delta l = \begin{cases} \nu\left(\left(\left(k-\frac{1}{2}\right)\Delta l,\ \left(k+\frac{1}{2}\right)\Delta l\right]\right), & k \neq 0, \\ 0, & k = 0, \end{cases} \tag{43}$$

and for all $k \in \mathbb{Z}$, $u \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, and a measurable set $A \subset \mathbb{R}$, set

$$\nu^{\mathrm{CP}}(A) = \sum_{k\Delta l \in A} \kappa_k \Delta l. \tag{44}$$

ν^{CP} is a discrete approximation to ν, concentrating the mass of ν in the set $\left(\left(k-\frac{1}{2}\right)\Delta l,\ \left(k+\frac{1}{2}\right)\Delta l\right]$ onto the point $\{k\Delta l\}$. L^{CP} is a process in continuous time with discete jump sizes. As $\Delta l \to 0$, $\nu^{\mathrm{CP}}(A) \equiv \nu^{\mathrm{CP}}_{\Delta l}(A) \to \nu(A)$ so $L^{\mathrm{CP}} \equiv L^{\mathrm{CP}}(\Delta l)$ converges to L. If ν has a Lévy density κ, κ_k approximates $\kappa(k\Delta l)$ and $\kappa_k \to \kappa(k\Delta l)$ as $\Delta l \to 0$.

By construction

$$\int_{\mathbb{R}\setminus\left(-\frac{1}{2},\frac{1}{2}\right]} \nu(\mathrm{d}x) = \int_{\mathbb{R}\setminus\left(-\frac{1}{2},\frac{1}{2}\right]} \nu^{\mathrm{CP}}(\mathrm{d}x) = \sum_{k\in\mathbb{Z}} \kappa_k \Delta l \equiv \lambda < \infty. \tag{45}$$

Now consider an approximatin $L^{\mathrm{CP}}_{\Delta t}$ to L^{CP} over a time step Δt. The generating triple of the variable $\Delta L_t = L_{\Delta t}$ is $(a\Delta t, \nu\Delta t, \gamma\Delta t)$. Define

$$\kappa_k^{\Delta t}\Delta l = \kappa_k \Delta t \Delta l = \begin{cases} \nu\left(\left(\left(k-\frac{1}{2}\right)\Delta l,\ \left(k+\frac{1}{2}\right)\Delta l\right]\right)\Delta t, & k \neq 0, \\ 0, & k = 0. \end{cases} \tag{46}$$

The Lévy measure $\nu^{\mathrm{CP}}_{\Delta t}$, with $\nu^{\mathrm{CP}}_{\Delta t}(A) = \sum_{k\Delta l\in A} \kappa_k^{\Delta t}\Delta l$ for a measureable set $A \subset \mathbb{R}$, is an approximation to $\nu\Delta t$ and converges to $\nu\Delta t$ as $\Delta l \to 0$. A discrete time approximation $L^{\mathrm{CP}}_{\Delta t}$ to L^{CP} is then

$$\Delta L^{\mathrm{CP}}_t = \Delta b_t + \Delta c_t, \tag{47}$$

$$\Delta b_t = \gamma\Delta t + \sqrt{a}\Delta z_t, \tag{48}$$

$$\Delta c_t = J\Delta N_t, \tag{49}$$

where z_t is a standard Brownian motion, N_t is a counting process with finite intensity λ and J is a random jump size constrained to jump sizes $k\Delta l$, $k \in \mathbb{Z}\setminus\{0\}$ Jump probabilities for a jump in J of size $k\Delta l$ are

$$p^J_k = \Pr[J = k\Delta l] = \begin{cases} \frac{\kappa_k^{\Delta t}\Delta l}{\lambda\Delta t}, & k \neq 0, \\ 0, & k = 0. \end{cases} \tag{50}$$

This is a discrete time approximation where J has discrete jump sizes that are multiples of Δl but b_t takes a continuous range of values.

To obtain a lattice approximation to L we modify the definitions just given. We approximate b_t and c_t by lattice processes $\widetilde{b}_t$ and $\widetilde{c}_t$.

For $\widetilde{c}_t$ we need only truncate the range of jump sizes. Let $k \in \left\{-D^J, \ldots, U^J\right\} = K^J$ be an indexing set. Define branching probabilities p_k^c by

$$p_k^c = \frac{\kappa_k^{\Delta t} \Delta l}{\widehat{\lambda} \Delta t}, \ k \in K^J, \tag{51}$$

where $\widehat{\lambda} = \sum_{k \in K^J} \kappa_k^{\Delta t} \Delta l$ is a normalising constant. To define $K^J \equiv K^J(\Delta t)$ we can choose U^J and D^J so that $\sum_{k > U^J} p_k^c < \frac{1}{2}\varepsilon$, $\sum_{k < D^J} p_k^c < \frac{1}{2}\varepsilon$ for some $\varepsilon > 0$.

To define a lattice approximation for b_t we can match moments. Set $P^b = \left\{p_k^b\right\}$ where $p_k^b = \Pr\left[\Delta \widetilde{b}_t = k\Delta l\right]$ for $k \in \left\{-D^b, \ldots, U^b\right\} = K^b$. Set $\Delta z_t = \frac{\Delta b_t - \gamma \Delta t}{a}$, then Δz_t is a standard Brownian motion with odd moments zero and even moments

$$\mathbb{E}\left[(\Delta z_t)^{2n}\right] = \frac{(2n)!}{2^n n!} (\Delta t)^n. \tag{52}$$

On the lattice we then require

$$\sum_{k \in K^b} p_k^b \left(\frac{k\Delta l - \gamma \Delta t}{a}\right)^{2n} = \frac{(2n)!}{2^n n!} (\Delta t)^n, \ n = 1, \ldots, D^b + U^b. \tag{53}$$

It is convenient to set $\Delta l = \kappa \gamma \Delta t$ for some constant κ, then

$$\sum_{k \in K^b} (k\kappa - 1)^{2n} p_k^b = \left(\frac{a}{\gamma}\right)^{2n} \frac{(2n)!}{2^n n!} (\Delta t)^{-n}, \ n = 1, \ldots, L^b + U^b. \tag{54}$$

The system (54) can be solved for $\left\{p_k^b\right\}_{k \in K^b}$. For a standard Brownian motion z_t Alford and Webber (01) [4] used $K^b = \{-3, -2, -1, 0, 1, 2, 3\}$ to match the first nine moments of z_t.

Now set $K = K^J \cup K^b$ and define $p_k = (1 - \lambda \Delta t) p_k^b + \lambda \Delta t p_k^c$ where $p_k^b = 0$ for $k \notin K^b$, $p_k^c = 0$ for $k \notin K^J$. This constructs a set of branching probabilities for the process $\widetilde{L}_t$, based on the representation (47), that converges to L_t as $(\Delta t, \Delta l) \to 0$.

Part of the Lévy densities of VG and NIG processes are shown in figure 3. The densities are unbounded near zero and have mass into the tails so that large jumps have a relatively large chance of occurring.

We did not find it possible to recover acceptable branching probabilities using the approach of this section applied to either the VG process or the NIG processs. However, for Lévy processes whose Lévy measure does not become arbitrarily large at zero, that is, for compound Poisson processes, this procedure may result in acceptable branching probabilities. This is useful, since although a compound Poisson process is relatively easy to simulate, a lattice implementation may enable American and Bermudan style options to be valued.

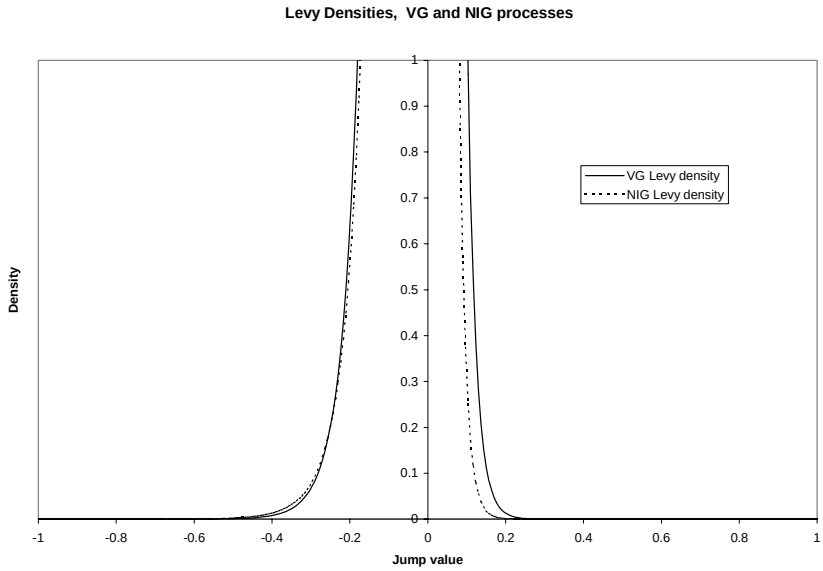

Figure 3: The VG and NIG Lévy Densities

3.2.2 Construction from a subordinated Brownian motion representation

Suppose a Lévy process is represented as a subordinated Brownian motion, $L_t = w_{h(t)}$ for a Brownian motion w_t and a stochastic subordinator $h(t)$. We construct branching probabilities for L on a grid with spacing $(\Delta t, \Delta l)$ by looking at the density of the subordinator h with the same space step Δl but with a refined time step $\widehat{\Delta t} < \Delta t$.

This construction may be most useful for Lévy processes defined primarily through a subordinator h. For instance in models of asset returns h could be obtained empirically as an estimate of 'market time'. In such cases it may not be suitable for numerical purposes to compute the transition density or the characteristic triple.

Let $H^{\Delta t}(x)$ be the distribution function of increments in h over a period Δt,

$$H^{\Delta t}(x) = \Pr\left[h(t+\Delta t) - h(t) \leq x\right]. \tag{55}$$

On a lattice with time step $\widehat{\Delta t} < \Delta t$ set

$$p_i^h = p_i^h\left(\widehat{\Delta t}\right) = H^{\Delta t}\left(i\widehat{\Delta t}\right) - H^{\Delta t}\left((i-1)\widehat{\Delta t}\right),\ i = 1, \ldots, \infty. \tag{56}$$

If $H^{\Delta t}$ has a density function $h^{\Delta t}(x)$, the probabilities p_i^h are a discrete approximation to $h^{\Delta t}\left(i\widehat{\Delta t}\right)\widehat{\Delta t}$.

Consider a lattice process $\widetilde{w}$ approximating a Brownian motion w, with $D = U = b$, with time and space step $\left(\widehat{\Delta t}, \Delta l\right)$ and branching probabilities $P = P\left(\widehat{\Delta t}, \Delta l\right) = \{p_k^w\}_{k=-b,\ldots,b}$. Here the set P may be found by some moment matching method. Let $p_{i,j}$ be the probability that $\widetilde{w}_{t_i}$ has value $\widetilde{w}_{i,j}$ at time $t_i = i\widehat{\Delta t}$,

$$p_{i,j} = \Pr\left[\widetilde{w}_{t_i} = \widetilde{w}_{i,j} \mid \widetilde{w}_{t_0} = \widetilde{w}_{0,0}\right]. \tag{57}$$

The values of $p_{i,j}$ can be computed iteratively from $p_{i-1,j}$, since

$$p_{i,j} = \sum_{k=-b}^{b} p_{i-1,j-k} p_k^w, \text{ for } |j-k| \leq (i-1)\, b, \tag{58}$$

$$p_{0,0} = 1. \tag{59}$$

We set

$$p_k^l = \Pr\left[\widetilde{l}_{t_{i+1}} = \widetilde{l}_{i+1,j+k} \mid \widetilde{l}_{t_i} = \widetilde{l}_{i,j}\right] = \sum_{q=\left\lceil \frac{k}{b} \right\rceil}^{\infty} p_{q,k} p_q^h. \tag{60}$$

When $b = 1$ the summation is from k to ∞. (60) is a discrete version of the result that a subordinated Brownian motion has marginal distributions that can be represented as a mixture of normals.

When h is deterministic we have

$$p_i^h = p_i^h\left(\widehat{\Delta t}\right) = \begin{cases} 1, & i\widehat{\Delta t} = \Delta h\,(t)\,, \\ 0, & \text{otherwise,} \end{cases} \tag{61}$$

so that

$$p_k^l = p_{i,k},\ i\widehat{\Delta t} = \Delta h\,(t)\,. \tag{62}$$

In practise the system (60) must be truncated at some level. It is convenient to do so at a level k^ε at which $1 - H^{\Delta t}\,(k\Delta t) < \varepsilon$, for some ε, setting $p_{k^\varepsilon}^h = 1 - H^{\Delta t}\,(k^\varepsilon \Delta t)$, and $p_k^h = 0$ for all $k > k^\varepsilon$.

Since the matrix P^w is fixed, the vector p^h completely determines the branching probabilities p^l. Conversely, the uniqueness of the Brownian subordination representation means that p^h is uniquely determined from p^l, in the limit.

In practice there can be difficulties in applying this procedure to finding branching probabilities. The VG and NIG subordinator densities for $\Delta t = 0.01$ are plotted in figure 4. The subordinator densities for $\Delta t = 0.01$ are unbounded at zero. For the VG subordinator over 30% of the probability mass in the interval (0, 1e-10]. This presents severe difficulties for the algorithm. Even taking $\widehat{\Delta t}$ very small - O (1e-15) - we were unable to match the moments of the VG process to any accuracy. The time taken to obtain the branching probabilities was correspondingly large.

The approach of this section could be useful when the subordinator was not unbounded near zero. For instance, for empirical work with 'market time' it might be more tractable to use a subordinator with this property over the time scales of interest.

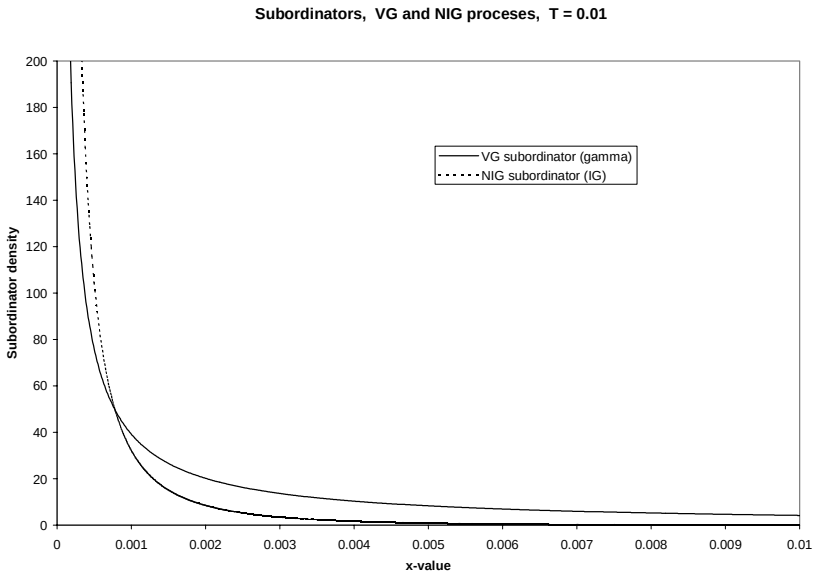

Figure 4: The VG and NIG subordinators, $\Delta t = 0.01$

3.2.3 Construction from a time copula

It is possible to construct a lattice for an arbitrary time-copula with arbitrary marginals. The procedure is

1. Construct a lattice with the specified marginal distributions.

2. Impose probabilities on to the lattice to endow it with the specified copula.

The usefulness of this construction is that it enables the marginals to be specified separately from the time copula. The marginals can be varied to fit, day by day, to market smile data, while keeping the time-copula fixed.

Suppose that a lattice process $\widetilde{w}$ approximating a continuous process w has values $\{\widetilde{w}_{i,j}\}$ at nodes (i,j) and that the marginal distributions $F^w_{i,j} = \Pr\left[\widetilde{w}_{t_i} \leq \widetilde{w}_{i,j} \mid \widetilde{w}_{t_0} = \widetilde{w}_{0,0}\right]$ are given. Given a copula $C(u,v)$ we can constuct branching probabilities on the lattice consistent with $F^w_{i,j}$ and with C.

Set $p_{i,j} = \Pr\left[\widetilde{w}_{t_i} = \widetilde{w}_{i,j} \mid \widetilde{w}_{t_0} = \widetilde{w}_{0,0}\right]$ so that

$$F^w_{i,j} = \Pr\left[\widetilde{w}_{t_i} \leq \widetilde{w}_{i,j} \mid \widetilde{w}_{t_0} = \widetilde{w}_{0,0}\right] = \sum_{k=-\infty}^{j} p_{i,k}. \tag{63}$$

We construct branching probabilities

$$p^w_{i,i+1}(j,k) = \Pr\left[\widetilde{w}_{t_{i+1}} = \widetilde{w}_{i+1,k} \mid \widetilde{w}_{t_i} = \widetilde{w}_{i,j}\right] \tag{64}$$

consistent with $F^w_{i,j}$ and yielding the desired time copula. The joint distribution function for $\widetilde{w}_{t_i}$ and $\widetilde{w}_{t_{i+1}}$ is

$$F^{J,w}_{i,i+1}(j,k) = \Pr\left[\widetilde{w}_{t_i} \leq \widetilde{w}_{i,j}, \widetilde{w}_{t_{i+1}} \leq \widetilde{w}_{i+1,k} \mid \widetilde{w}_{t_0} = \widetilde{w}_{0,0}\right], \tag{65}$$

$$= \sum_{r=-\infty}^{j} p_{i,r} \sum_{s=-\infty}^{k} p^w_{i,i+1}(r,s). \tag{66}$$

When w_t is a Lévy process the branching probabilities obey $p^w_{i,i+1}(j,k) \equiv p^w_{i,i+1}(j-k)$.

Given $F^{J,w}_{i,i+1}(j,k)$ the branching probabilities $p^w_{i,i+1}$ can be recovered

$$p_{i,j}p^w_{i,i+1}(j,k) = \Pr\left[\widetilde{w}_{t_{i+1}} = \widetilde{w}_{i+1,k} \mid \widetilde{w}_{t_i} = \widetilde{w}_{i,j}\right] \tag{67}$$

$$= F^{J,w}_{i,i+1}(j,k) - F^{J,w}_{i,i+1}(j,k-1) - F^{J,w}_{i,i+1}(j-1,k) + F^{J,w}_{i,i+1}(j-1,k-1). \tag{68}$$

But from the copula $C(u,v)$ we can define

$$F^{J,w}_{i,i+1}(j,k) = C\left(F^w_{i,j}, F^w_{i+1,k}\right). \tag{69}$$

Hence, by (68), we can extract branching probabilities consistent with the marginal distributions $F^w_{i,j}$ and the copula, C.

Conversely given branching probabilities $p^w_{i,i+1}(j,k)$ one obtains $F^w_{i,i+1}(j,k)$ from (66) and hence $C(u,v)$ from (69). C is defined only at points $\left(F^w_{i,j}, F^w_{i+1,k}\right)$.

This construction is very general. In financial applications, conditions need to be imposed to ensure that the lattice process $\widetilde{w}_{t_i}$ is a martingale.

3.3 Implied Pricing Methods

Implied lattices and grids are used to ensure that prices observed in the market are recovered exactly by the lattice. Popular methods originate with Dupire (94) [24] and Rubinstein (94) [45]. An implied grid method is described by Andersen and Brotherton-Ratcliffe (97) [6]. However, these methods implicitly assume that the underlying state variable follows a diffusion process. If prices in the market are determined by state variables following Lévy processes these methods will fail. For instance, several implied lattices or implied grids assume either a trinomial branching for the state variable in the lattice, or that prices are propagated through the lattice using an implicit or Crank-Nicolson finite difference scheme. The latter schemes arise from solving numerically the Feynman-Kac equation - based upon a diffusion scheme.

An alternative to using a method of this sort is to use a grid to back out a time copula from market prices. Time copulas are not constrained to be determined from underlying diffusions but can emerge from arbitrary Lévy processes. There are two ideas:

1. Given sufficiently dense data, back out the empirical implied time copula from market prices.

2. If there is insufficient data to do (1), then impose a given time-copula on the data.

When there is insufficient data then imposing a time-copula becomes an interpolation/extrapolation exercise - possibly an attractive alternative to other methods.

In principle the method proceeds as follows. Firstly, extract marginal densities from market data at times for which suitable data is available. For instance, suppose that series of European calls mature at times $t_1, \ldots, t_n$. Using standard methods obtain probabilities $p_{i,j}$ for times t_i, corresponding to equation (63). If no other data is available, a copula - for instance a Gaussian time copula - can be imposed between successive times (t_i, t_{i+1}) to create a lattice scheme, by the procedure outlined above in equations (69) and (66). The chosen time copula can be used to branch to intermediate times $t_{i+\frac{1}{2}}$ if required, although a suitable marginal distribution would then also have to be applied at the intermediate time.

If instruments are available whose value depend, explicitly or implicitly, upon the values of the underlying at serval different times, these can be used either to

1. Calibrate the chosen time copula or
2. To back out the market time copula

4 Applications and Numerical Results

In this section we apply the Lévy lattice pricing method to the VG and NIG processes. In practice lattice methods can be speeded up by using a variety of special methods. These include incorporating a 'terminal correction' to obtain faster, uniform convergence, and the use of Richardson extrapolation, and numerical integration can be improved using a quadrature method such as Simpson's rule. We do not employ these methods here.

Benchmarking to European Calls We use the lattice to price European and Bermudan call options for each model using our standard parameter sets. The options have time to maturity one year and strikes ranging from 90 to 120, with an initial stock value $S_0 = 100$. We compute option prices on the lattice for various grid refinements and branching specifications. The implementation was Matlab 5.3 on a 500 Mhz PC, using built-in Matlab functions to compute Bessel functions. The code used no special speed-ups. Since only one option price was required from the lattice on each run, the number of nodes computed at each time step, working back from the final time, was successively reduced to minimise the computation. Times are in seconds.

Table 3 gives results from the lattice for the VG process using ten time steps over a one year period. The space step Δl and the number of space steps M were set so that extreme returns on the lattice ranged from -400% to $+400\%$

		Analytic	**Lévy Lattice**		
	M		4000	1000	400
	Δl		0.001	0.005	0.010
	U		500	200	50
	D		1000	400	50
	90	19.09935	19.09936	19.09994	19.10669
	95	15.07047	15.07048	15.07111	15.07678
	100	11.37003	11.37002	11.37048	11.37462
X	105	8.11978	8.11978	8.12040	8.12401
	110	5.42960	5.42960	5.42972	5.43375
	115	3.36543	3.36544	3.36571	3.36639
	120	1.92110	1.92110	1.92126	1.92114
time		0.3	204.2	21.9	4.3

Table 3: European call prices: Variance-gamma lattice. T=1, N=10

		Reference	**Lévy Lattice**		
	M		4000	1000	400
	Δl		0.001	0.005	0.010
	U		500	200	50
	D		1000	400	50
	90	19.09330	19.09331	19.09333	19.09903
	95	15.06077	15.06078	15.06089	15.06521
	100	11.35994	11.35993	11.35994	11.36259
X	105	8.11561	8.11561	8.11586	8.11810
	110	5.43723	5.43724	5.43709	5.44004
	115	3.38474	3.38475	3.38485	3.38488
	120	1.94359	1.94359	1.94367	1.94328
time		2.6	194.7	20.1	4.5

Table 4: European call prices: NIG lattice. T=1, N=10

over the year. The number of up and down branches, U and D were fixed so that branching over a time step of 0.1 covered a returns range of $\pm 50\%$. In this illustration we took $k_q = q$. Times in seconds are for computing all seven option values.

We see that with a time step of this size the lattice is capable of reproducing benchmark call prices as long as the lattice is fine enough. With a course lattice in the money calls are priced less accurately than out of the money calls. Table 4 gives comparable results for the NIG process. Recall that parameter values have been chosen so that the first four moments are matched to those of the VG process. Reference values, computed by numerical integration as before, are greater than variance gamma prices for out of the money calls, but smaller for in the money calls. The lattice behaves similarly for the NIG and the VG cases.

		Lévy Lattice		
	M	4000	1000	400
	Δl	0.001	0.005	0.010
	U	500	200	50
	D	1000	400	100
	90	0.76115	0.76113	0.75739
	95	1.52574	1.52579	1.52181
	100	2.88152	2.88140	2.87784
X	105	5.17036	5.17027	5.16808
	110	9.04064	9.04049	9.03801
	115	13.87623	13.87614	13.87525
	120	18.80965	18.80960	18.80859
time		188.8	20.0	4.3

Table 5: Bermudan put prices: VG lattice. T=1, N=10

		Lévy Lattice		
	M	4000	1000	400
	Δl	0.001	0.005	0.010
	U	500	200	50
	D	1000	400	100
	90	0.74482	0.74488	0.73793
	95	1.49554	1.49563	1.48796
	100	2.84445	2.84453	2.83763
X	105	5.17297	5.17311	5.16859
	110	9.03395	9.03367	9.02986
	115	13.86529	13.86525	13.86415
	120	18.80693	18.80693	18.80574
time		202	24.7	4.7

Table 6: Bermudan put prices: NIG lattice. T=1, N=10

Valuing Bermudan Puts We now use the lattice to price a Bermudan put that can be exercised at time $t = i/10$, $i = 1, \ldots 10$. Results for the VG process are given in table 5. The lattice gives prices that appear to be accurate to 2dp at the finest resolution.

For the NIG process we obtain the results shown in table 6, with apparently similar accuracy.

We now investigate using the lattice with a small time step. To fit the density close to its singularity at zero, we use the scheme described in equation (42). Table 7 values a VG option using 20 time steps over a time to maturity of 0.2 years, so that each time step is 0.01. The $(p, k) = (3, 2)$ specification seems superior to the $(p, k) = (2, 2)$ specification. Ordinary branching with $(p, k) = (1, 1)$ (not reported) gives less accurate results than either of the other two. Despite the small time step the most refined lattice benchmarks very well

		Analytic	Lévy Lattice			
	M		4000	2000	1000	400
	Δl		0.001	0.001	0.001	0.01
	U		500	400	400	50
	D		1000	800	800	100
	(p,k)		$(2,3)$	$(2,2)$	$(2,2)$	$(2,3)$
X	90	11.97158	11.97146	11.95967	11.84315	11.97128
	95	7.42099	7.42091	7.41482	7.35141	7.42089
	100	3.43796	3.43791	3.43378	3.40146	3.44360
	105	0.82924	0.82924	0.82639	0.80948	0.83273
	110	0.14962	0.14955	0.14790	0.14112	0.14740
	115	0.02913	0.02909	0.02836	0.02638	0.02908
	120	0.00608	0.00605	0.00582	0.00527	0.00606
time		0.4	425.5	187.9	81.0	4.9

Table 7: European call values: VG, T = 0.2, N = 20

to the analytic option values. Even the coarsest specification with $(p,k) = (3,2)$ is surprisingly accurate.

Table 8 gives results for the NIG process. Here also results are very accurate, although compared to the VG case errors are greater at the coarsest resolution.

Valuing American Puts We investigate the pricing of an American put option maturing in $T = 0.2$ years. Tables 9 and 10 report VG and NIG option prices with successive more exercise opportunities up to final maturity. N is the number of exercise opportunities, for exercise at times $\frac{1}{N}T, \frac{2}{N}T, \ldots, T$. The option cannot be exercised immediately, at time zero. One time step was taken between each exercise opportunity. The tables report the lattice specification used in each case.[12]

For neither process have the options values converged to their American limit at 20 exercise opportunities.

5 Conclusions

In this paper we have discussed lattice implementations of Lévy processes using the density of the Lévy process. We discussed alternative constructions through the Brownian subordinator, through the generating triple, or from the time copula. We have shown how lattice methods for Lévy processes may be developed and have benchmarked our implementations to the VG and NIG based models. We have used the lattice to price Bermudan and American options in these models, presenting numerical results. Not reported in this paper, we also

[12] Option values for $N = 40$ may be questionable: we note that for out of the money strikes the lattices gives lower option values for $N = 40$ than for $N = 20$.

		Reference	Lévy Lattice			
	M		4000	2000	1000	400
	Δl		0.001	0.001	0.001	0.010
	U		500	400	400	50
	D		1000	800	800	100
	(p,k)		$(2,3)$	$(2,2)$	$(2,2)$	$(2,3)$
X	90	11.96191	11.96192	11.96192	11.96192	11.95922
	95	7.39739	7.39740	7.39740	7.39740	7.38837
	100	3.43925	3.43927	3.43926	3.43926	3.41297
	105	0.91070	0.91075	0.91074	0.91074	0.87845
	110	0.12650	0.12650	0.12650	0.12650	0.12035
	115	0.01454	0.01455	0.01455	0.01455	0.01359
	120	0.00180	0.00180	0.00180	0.00180	0.00171
time		2.6	2245.8	984.7	520.0	60.8

Table 8: European call values: NIG, T = 0.2, N = 20

		Lévy Lattice					
	N	1	2	5	10	20	40
	M	4000	4000	4000	4000	4000	4000
	Δl	0.001	0.001	0.001	0.001	0.001	0.001
	U	1000	1000	1000	500	500	200
	D	2000	2000	2000	1000	1000	400
X	90	0.18948	0.20021	0.20808	0.21099	0.21250	0.21122
	95	0.53989	0.57557	0.59975	0.60836	0.61278	0.61277
	100	1.45783	1.57808	1.65234	1.67737	1.68994	1.69403
	105	3.75014	4.27371	4.67925	4.83437	4.91588	4.95752
	110	7.97146	8.94745	9.57153	9.78469	9.89209	9.94592
	115	12.75198	13.86260	14.54249	14.77086	14.88534	14.94259
	120	17.62992	18.80725	19.52122	19.76034	19.88010	19.93999
time		0.9	34.9	139.1	196.5	508.9	650.2

Table 9: Bermudan put option values: VG model, T = 0.2, N exercise opportunities. Convergence to American values.

		Lévy Lattice					
	N	1	2	5	10	20	40
	M	4000	4000	4000	4000	4000	4000
	Δl	0.001	0.001	0.001	0.001	0.001	0.001
	U	1000	1000	1000	500	500	200
	D	2000	2000	2000	1000	1000	400
	90	0.17980	0.18913	0.19618	0.19885	0.20025	0.19683
	95	0.51627	0.54760	0.56964	0.57776	0.58200	0.57954
	100	1.45912	1.57657	1.64912	1.67412	1.68689	1.68890
X	105	3.83159	4.32543	4.68518	4.83072	4.91213	4.95507
	110	7.94835	8.92966	9.56495	9.78168	9.89067	9.94516
	115	12.73739	13.85781	14.54125	14.77035	14.88511	14.94240
	120	17.62564	18.80621	19.52099	19.76025	19.88007	19.93989
time		0.8	34.3	136.3	172.1	353.6	527.9

Table 10: Bermudan put option values: NIG model, T = 0.2, N exercise opportunities. Convergence to American values.

applied the lattice to the Meixner process (Schoutens (02) [49]) finding good results.

Using the density function representation is was possible to obtain branching probabilities for all the processes we investigated, enabling us to prices Bermudan style options, and to obtain bounds on American style options.

Because of the behaviour of their Lévy densities near zero, we were not able to obtain a lattice for any of these process using the generating triple representation. Over short time horizons we found difficulties in using the Brownian subordinator representation with the VG and NIG processes as their subordinator densities become unbounded near zero.

We have demonstrated that lattice methods may be used in conjunction with Lévy processes, particularly in cases where explicit solutions for option prices may not exist. These methods may open up the prospect of the greater use of Lévy process based models in the future.

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