Approximate cores of games and economies with clubs.[¤]

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Abstract

We introduce the framework of parameterized collections of games with and without sidepayments and provide three nonemptiness of approximate core theorems for games in parameterized collections. The parameters bound (a) the

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number of approximate types of players and the size of the approximation and (b) the size of nearly e®ective groups of players and their distance from exact e®ectiveness. The theorems are based on a new notion of partition-balanced pro¯les and approximately partition-balanced pro¯les. The results are applied to a new model of an economy with clubs. In contrast to the extant literature, our approach allows both widespread externalities and uniform results.

1   Introduction.

We introduce the notion of parameterized collections of games and show that, under apparently mild conditions, approximate cores of all su±ciently large games without side payments are nonempty. A collection of games is parameterized by (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly e®ective groups of players and their distance from exact e®ectiveness. All games described by the same parameters are members of the same collection. The conditions required on a parameterized collection of games to ensure nonemptiness of approximate cores are merely that most players have many close substitutes, per capita payo®s are bounded (per capita boundedness), and all or almost all gains to collective activities can be realized by groups bounded in size (small group e®ectiveness). Per capita boundedness simply rules out arbitrarily large average payo®. The ¯nal condition, small group e®ectiveness, may appear to be restrictive, but, in fact, in the context of a \pregame," if there are su±ciently many players of each type, then per capita boundedness and small group e®ectiveness are equivalent (Wooders 1994b).[1]

As an application of our research, we develop a new model of an economy with clubs and obtain analogues of our non-emptiness results for games. Following Buchanan (1962), Shubik and Wooders (1982) and other recent papers, we allow individuals to belong to multiple clubs. In contrast to prior research in this area, our model allows utilities from forming a club to be a®ected by the size and composition of the economy containing the club. For example, there may be widespread externalities.

To position our model and results in the literature, recall that Shapley and Shubik (1966) showed that large replica exchange economies with quasi-linear preferences have nonempty approximate cores. Under the assumption of per capita boundedness { ¯niteness of the supremum of average payo® { Wooders (1980,1983) demonstrated nonemptiness of approximate cores of large games with and without side payments. Since then, there have been a number of advances in this literature, including Shubik and Wooders (1983), Kaneko and Wooders (1982), and Wooders and Zame (1984). The prior literature on approximate cores of large games all uses the framework of

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[1] This equivalence requires that games under consideration have su¯ciently many players of all types that appear in the games. See also Wooders (1991) for related results for games without side payments.

a pregame. A pregame consists of a compact metric space of player types, possibly finite, and a worth function ascribing a payoff possibilities set to every possible group of players. The worth function depends continuously on the types of players in a coalition. The pregame framework treats collections of games that can all be described by a single worth function. This has hidden consequences; for example, as we will illustrate, the equivalence between small group effectiveness and per capita boundedness noted above depends on the structure of a pregame. Moreover, the pregame framework dictates that the payoff set of a coalition cannot depend on the total player set of the game in which it is embedded; widespread externalities are ruled out.[2]

To illustrate how parameterized collections can treat a broader class of situations than pregames, consider, for example, a sequence of economies where the nth economy has n identical players. Due to widespread negative externalities, in the nth economy each agent can realize a payoff of $1 + 1/n$. Also suppose, for simplicity, that in the nth economy, a coalition containing $m \cdot n$ can realize the total payoff of $m(1 + 1/n)$ { within each economy there are no gains to coalition formation. (It is easy to modify the example to allow such gains.) The pregame framework rules out such sequences of games. In contrast, parameterized collections of games incorporate games with widespread externalities and our results apply.[3] This example also illustrates that our club-theoretic results cannot be obtained in the pregame context.

Turning to the motivation for our research, it is well understood that except in highly idealized situations cores of games may be empty and competitive equilibrium of economies may not exist. For example, within the context of an exchange economy, the conditions required for existence of equilibrium typically include convexity, implying infinite divisibility of commodities, and also nonsatiation. Even these two conditions may well not be satisfied; goods are usually sold in pre-specified units and there are some commodities that many individuals prefer not to consume. In economies with congestion, even if preferences and production technologies are convex, the core of the economy may be empty.[4] In the context of economies with coalition structures, such as economies with clubs and/or local public goods, the added difficulties of endogenous group formation compound the problem; even if, given club memberships, all conditions for existence of equilibrium and nonemptiness of the core are satisfied in each club, the core may be empty. One possible approach to the problem of existence of equilibrium is to restrict attention to models where equilibria exist, for example, economies with continuums of agents. But a model with a continuum of agents can only be an approximation to a finite economy. Another

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[2]An exception is Wooders (1983), which allows positive externalities.

[3]In spirit, the pregame framework is similar to the economic frameworks of Kannai (1970) and Hildenbrand (1974), for example, while our approach is more in the spirit of the economic models of Anderson (1978) and Manelli (1991a,b).

[4]An example making this point appears in Shubik and Wooders (1983).

approach is to consider solution concepts for which existence is more robust, for example, approximate equilibria and cores. It seems reasonable to suppose that there are typically frictions that prevent attainment of an exact competitive equilibrium. At any time, most markets may have some unsatisfied demand or supply and most purchases might be made at prices that are only close to equilibrium prices. It also seems reasonable to suppose that there are typically costs of forming coalitions. These sorts of observations motivate the study of existence of approximate equilibria and nonemptiness of approximate cores.

Besides assumptions on the structure of the economies or games considered, solution theory also requires behavioral assumptions { the competitive equilibrium requires that individuals take prices as given (by some unknown source) and optimize while the core is based on the idea that if a group of individuals can be better off by forming a coalition and reallocating resources and activities within that coalition, then they will do so. These behavioral assumptions are problematic, those of the core perhaps no more so than those of the competitive equilibrium. An alternative to the behavioral assumption of the core that may be easier for economists to swallow is that entrepreneurs form coalitions whenever there exists an opportunity to profit from doing so; there are a number of papers in the literature taking this approach or closely related approaches, for example, Pauly (1967), Shapley and Shubik (1969), Wooders (1979) and Bennett and Wooders (1979).[5] The literature on contestable markets, for example, Baumol, Panzer and Willig (1982) takes a similar approach; roughly this literature suggests that the presence of entrepreneurs who are ever-ready to enter a market if there is an opportunity to profit ensures that prevailing prices are perfectly competitive. Thus, there is some motivation for both the core and the competitive equilibrium in the idea that if prices/payoffs are not competitive, profit maximizing firms will enter. From the viewpoint of the behavioral assumptions required, there is no apparent compelling reason to favor either the core or the competitive equilibrium.

Since the seminal papers of Shubik (1959), Debreu and Scarf (1963) and Aumann (1964), equivalence of the core and price-taking equilibrium and approximate equivalence has been shown in a variety of contexts, including in economies with coalition production (Hildenbrand (1974) and Böhm (1974) for example), in economies with public goods (Conley (1994) and Vasil'ev, Weber and Wiesmeth (1995) for example), economies with local public goods or clubs (several papers referenced herein { see Wooders (1999) for further references) and in economies with finite coalitions and widespread externalities (Hammond, Kaneko and Wooders (1989) and Hammond

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[5]Pauly treats the case of identical players and economies with quasi-transferable utilities. Wooders (1979) treats multiple types; her results are applied in Bennett and Wooders (1979) to problems of firm formation. For NTU games, it is easy to show that under certain conditions, the core is equivalent to a \no-entry" equilibrium; such results were obtained in Wooders (1982).

Shapley and Shubik (1969) made the connection between games with side payments and markets, defined as economies with quasi-linear preferences and concave utlity functions.

(1999) for example). Indeed, with quasi-linear utilities, large economies, including those with nonconvexities, indivisibilities, coalition production, public projects and clubs are equivalent to markets if and only if small groups are effective for the realization of all or almost all gains to coalition formation (Wooders 1994b). All these results suggest that in diverse large economies, approximate cores are nonempty and close to competitive outcomes. They also all depend on structures of specific economic models. The study of cores of general large games allows us to demonstrate that certain properties of economies depend only on a few features of the model, most notably, small group effectiveness.[6]

Finally, the game theoretic environment we consider may be especially well suited to the core. Since the games treated have many players of each of a relatively few approximate types, informally, we might expect that some random sorting process and bargaining may lead to optimal outcomes. In the context of an exchange economy, results in this spirit have been obtained by a number of authors, most recently Dagan, Serrano and Volij (1996).

In the remainder of this introduction, we first discuss our game-theoretic framework and results in more detail and then discuss economies with clubs. Related literature is discussed in the body of the paper.

1.1   The game-theoretic model and results.

We provide three theorems showing non-emptiness of approximate cores of arbitrary games. Given the specification of an approximate core { the particular approximate core notion and the parameters describing the closeness of the approximation { we obtain a lower bound $\acute{}$ on the number of players so that any game in the class of games described by the specified parameters with at least $\acute{}$ players has a nonempty approximate core. While our three theorems each use different notions of approximate cores, both the notions of approximate cores and the theorems build on each other. Our framework encompasses games derived from pregames with or without side payments and our results encompass, as special cases, a number of nonemptiness of approximate core results in the literature. In the concluding section of the current paper we remark on other applications of the notion of parametrized collections of games.

Our first result, for the "-remainder core, requires a finite integer number T of types of players and a bound B on effective group sizes. Roughly, a payoff vector is in the "-remainder core if it is in the core of a subgame containing all but a fraction " of the players. The result provides a lower bound, depending on T; B; and ", on the number of players required to ensure nonemptiness of the "-remainder core for all games with T types and bound B on effective group sizes: An important aspect

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[6]This has been the motivation of a continuing line of research on large games. See, for example, the discussions in Wooders (1979,1983,1994a,b,1999).

of this result, like the result of Kaneko and Wooders (1982), is that the conclusion is independent of the payoff sets for the games. The result is eminently applicable to models with bounded coalition sizes, such as marriage and matching games (cf., Kelso and Crawford (1982) or Roth and Sotomayor (1992)).

The ε-core of a game is the set of feasible payoff vectors that cannot be improved upon by any coalition of players by at least ε for each member of the coalition. A payoff vectors is in the ε_1-remainder ε_2-core if it induces a payoff vectors in the ε_2-core of a subgame containing all but a fraction ε_1 of the players. Our non-emptiness theorem for the ε_1-remainder ε_2-core requires only that groups bounded in size are effective for the realization of almost all gains to cooperation. Instead of the assumption of a finite number of types, to show non-emptiness of the ε_1-remainder ε_2-core we require only that there be a partition of the set of players into a finite number of approximate types. Such an assumption would be satisfied by games derived from a pregame with a compact metric space of player types, for example.

Under two additional restrictions on the class of games, we obtain a nonemptiness result for ε-cores. The restrictions are that: (a) per capita payoffs are bounded; and (b) the games are strongly comprehensive (that is, the boundaries of the total payoff set are bounded away from being "flat"). A corollary relaxes assumption (b).

1.2 Economies with clubs.

There are now numerous papers in the literature studying cores and equilibria of economies with local public goods, where a feasible state of the economy includes a partition of the set of agents into disjoint jurisdictions or clubs for the purposes of collective consumption of public goods within each club or jurisdiction.[7] There have been far fewer works on economies where an agent can belong to multiple clubs. In this paper we develop a model of an economy with clubs where: (a) an agent may belong to multiple clubs { indeed, as many clubs as there are groups containing that agent; (b) all agents may differ from each other; (c) each club may provide a unique bundle of goods and/or services, including private goods, public goods subject to exclusion, and conviviality; and (d) the payoff set of a club may depend on the economy in which it is embedded { widespread externalities are permitted.

A club is a group of people who collectively consume and/or produce a bundle of goods and/or services for the members of the club. Often clubs have been treated as synonymous with coalitions of agents providing congestable and excludable public goods for their members. We observe, however, that clubs engage in a variety of activities. These activities may or may not require input of private goods. The goods provided by the club may include the enjoyment of the company of the other club members. In clubs of intellectuals, the exchange of ideas may be the aspect of the

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[7]Some early papers include, for example, Wooders (1978) and Greenberg and Weber (1986). See Conley and Wooders (1995) and Konishi, Le Breton, and Weber (1998) for more recent references.

club that brings enjoyment to its members. Clubs may provide only private goods; for example, many academic departments have co®ee clubs. Other clubs o®er some goods and/or services to the general public. Some sorts of clubs o®er private goods and/or services to their members in addition to public goods. There is frequently no requirement that members of the same club consume the same bundles of goods. Thus, in this paper for each club we assume that there is an abstract set of feasible club activities.[8]

It may be the case that some sorts of clubs are ruled out for legal, technical, or social reasons. For example, a marriage may be viewed as a club, and polyandrous marriages may be illegal. Thus, for each coalition of agents in the economy there is an admissible club structure of that coalition. Admissible club structures are required to satisfy certain natural properties. In addition our model is required to satisfy the conditions that: (a) average utilities are bounded independently of the size of the economy; and (b) as the economy grows large, there is a limit to increasing returns to club size.

Although the conditions on our model are remarkably non-restrictive, by application of our game-theoretic results we are able to show several forms of the result that approximate cores of large economies { with su±ciently many players { are non-empty. Our result applies simultaneously to all games in a parameterized collection.

1.3 Organization of the paper.

The paper is organized as follows. The next section introduces the basic de¯nitions, including the notion of parametrized collections of games. Section 3 presents our three theorems on non-emptiness of approximate cores in the order presented above. Section 4 consists of our club model and results. Section 5 presents the mathematical foundation and a new mathematical result on approximate balancedness of large pro¯les of player sets. Section 6 concludes the body of the paper. Appendix 1 contains the proofs and Appendix 2 presents some additional examples, illuminating the mathematics of our results.

2   De¯nitions.

2.1 Cooperative games: description and notation.

Let $N = f1; ::::; ng$ denote a set of players. A non-empty subset of N is called a coalition. For any coalition S let R^S denote the jSj-dimensional Euclidean space with coordinates indexed by elements of S. For $x \ 2 \ R^N$; x_S will denote its restriction to R^S. To order vectors in R^S we use the symbols $>>$; $>$ and $\cdot$ with their usual

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[8]The notion of public projects, introduced in Mas-Colell (1980) and extended to local public projects in Manning (1992) and a number of subsequent papers, is related.

interpretations. The non-negative orthant of $\mathbb{R}^S$ is denoted by $\mathbb{R}^S_+$ and the strictly positive orthant by $\mathbb{R}^S_{++}$. We denote by 1_S the vector of ones in $\mathbb{R}^S$, that is, $1_S = (1,\ldots,1) \in \mathbb{R}^S$. Each coalition S has a feasible set of payoff or utility vectors denoted by $V_S \subseteq \mathbb{R}^S$. By agreement, $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is non-empty, closed and bounded from above for any i. In addition, we will assume that

$$\max\left\{x : x \in V_{\{i\}}\right\} = 0 \text{ for any } i \in N,$$

this is by no means restrictive since it can always be achieved by a normalization.

It is convenient to describe the feasible payoff vectors of a coalition as a subset of $\mathbb{R}^N$. For each coalition S let $V(S)$, called the payoff set for S, be defined by

$$V(S) := \left\{x \in \mathbb{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

A game without side payments (called also an NTU game or simply a game) is a pair (N, V) where the correspondence $V : 2^N \multimap \mathbb{R}^N$ is such that $V(S) \subseteq \left\{x \in \mathbb{R}^N : x_a = 0 \text{ for } a \notin S\right\}$ for any $S \subseteq N$ and satisfies the following properties :

(2.1) $V(S)$ is non-empty and closed for all $S \subseteq N$.

(2.2) $V(S) \setminus \mathbb{R}^N_+$ is bounded for all $S \subseteq N$, in the sense that there is a real number $K > 0$ such that if $x \in V(S) \setminus \mathbb{R}^N_+$, then $x_i \cdot K$ for all $i \in S$.

(2.3) $V(S_1 \cup S_2) \supseteq V(S_1) + V(S_2)$ for any disjoint $S_1, S_2 \subseteq N$ (superadditivity).

We next introduce the uniform version of strong comprehensiveness assumed for our third approximate core result. Roughly, this notion dictates that payoff sets are both comprehensive and uniformly bounded away from having level segments in their boundaries. Consider a set $W \subseteq \mathbb{R}^S$. We say that W is comprehensive if $x \in W$ and $y \cdot x$ implies $y \in W$. The set W is strongly comprehensive if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$:[9] Given (i) $x \in \mathbb{R}^S$, (ii) $i, j \in S$, (iii) $0 \cdot q \cdot 1$ and (iv) $\varepsilon \geq 0$, define a vector $x^q_{i,j}(\varepsilon) \in \mathbb{R}^S$, where

$$
\begin{aligned}
(x^q_{i,j}(\varepsilon))_i &= x_i - \varepsilon, \\
(x^q_{i,j}(\varepsilon))_j &= x_j + q\varepsilon, \text{ and} \\
(x^q_{i,j}(\varepsilon))_k &= x_k \text{ for } k \in S \setminus \{i,j\}.
\end{aligned}
$$

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[9]Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. This property has also been called \nonleveledness."

The set W is q-comprehensive if W is comprehensive and if, for any $x \in W$, it holds that $(x_{i,j}^q(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \geq 0$.[10] This condition for $q > 0$ uniformly bounds the slopes of the Pareto frontier of payoff sets away from zero. Note that for $q = 0$, 0-comprehensiveness is simply comprehensiveness. Also note that if a game is q-comprehensive for some $q > 0$ then the game is q'-comprehensive for all q' with $0 \leq q' \leq q$:

Let $V_S \subseteq R^S$ be a payoff set for $S \subseteq N$: Given q, $0 \leq q \leq 1$; let $W_S^q \subseteq R^S$ be the smallest q-comprehensive set that includes the set V_S.[11] For $V(S) \subseteq R^N$ let us define the set $c_q(V(S))$ in the following way:

$$c_q(V(S)) := \left\{ x \in R^N : x_S \in W_S^q \text{ and } x_a = 0 \text{ for } a \notin S \right\}:$$

Notice that for the relevant components { those assigned to the members of S { the set $c_q(V(S))$ is q-comprehensive, but not for other components. With some abuse of the terminology, we will call this set the q-comprehensive cover of $V(S)$: When $q > 0$ we can think of a game as having some degree of \side-paymentness" or as allowing transfers between players, but not necessarily at a one-to-one rate. This is an eminently reasonable assumption for games derived from economic models.

A game with side payments (also called a TU game) is a game (N, V) with 1-comprehensive payoff sets, that is $V(S) = c_1(V(S))$ for any $S \subseteq N$: This implies that for any $S \subseteq N$ there exists a real number $v(S) \geq 0$ such that $V_S = \left\{ x \in R^S : \sum_{i \in S} x_i \leq v(S) \right\}$. The numbers $v(S)$ for $S \subseteq N$ determine a function v mapping the subsets of N to R_+. Then the TU game is represented as the pair (N, v).

2.2 Parameterized collections of games.

To introduce the notion of parameterized collections of games we will need the concept of Hausdorff distance. For every two non-empty subsets E and F of a metric space (M, d); define the Hausdorff distance between E and F (with respect to the metric d on M), denoted by $dist(E, F)$, as

$$dist(E, F) := \inf \{ \varepsilon \in (0, 1) : E \subseteq B_\varepsilon(F) \text{ and } F \subseteq B_\varepsilon(E) \};$$

where $B_\varepsilon(E) := \{ x \in M : d(x, E) \leq \varepsilon \}$ denotes an ε-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance

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[10]The notion of q-comprehensiveness can be found in Kaneko and Wooders (1996). For the purposes of the current paper, q-comprehensiveness can be relaxed outside the individually rational payoff sets.

[11]Notice that there exist q-comprehensive sets that contain V_S, specifically R^S: The set W_S^q is the intersection of all q-comprehensive sets containing V_S.

between the intersection of the sets and a subset of Euclidean space. Let m^π be a ¯xed positive real number. Let M^π be a subset of Euclidean space R^N de¯ned by $M^\pi := \{x \in R^N : x_a \geq m^\pi \text{ for any } a \in N\}$. For every two non-empty subsets E and F of Euclidean space R^N let $H_1[E;F]$ denote the Hausdor® distance between $E \setminus M^\pi$ and $F \setminus M^\pi$ with respect to the metric $\|x - y\|_1 := \max_i |x_i - y_i|$ on Euclidean space R^N.

The concepts de¯ned below lead to the de¯nition of parameterized collections of games. To motivate the concepts, each is related to analogous concepts in the pregame framework.

<u>±¡ substitute partitions:</u> In our approach we approximate games with many players, all of whom may be distinct, by games with ¯nite sets of player types. Observe that for a compact metric space of player types, given any real number $\pm > 0$ there is a partition (not necessarily unique) of the space of player types into a ¯nite number of subsets, each containing players who are \±-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a ¯nite number of approximate types.

Let $(N;V)$ be a game and let $\pm \geq 0$ be a non-negative real number. A ±-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are \within ±" of being substitutes for each other. Formally, given a set $W \subseteq R^N$ and a permutation ι of N, let $\frac{3}{4}_\iota(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation ι. Given a partition $\{N[t] : t = 1,\ldots,T\}$ of N, a permutation ι of N is type ¡ preserving if, for any $i \in N; \iota(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A ±-substitute partition of N is a partition $\{N[t] : t = 1,\ldots,T\}$ of N with the property that, for any type-preserving permutation ι and any coalition S,

$$H_1\left[V(S); \frac{3}{4}_\iota^{-1}(V(\iota(S)))\right] \cdot \pm.$$

Note that in general a ±-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

<u>(±,T)- type games.</u> The notion of a (±,T)-type game is an extension of the notion of a game with a ¯nite number of types to a game with approximate types.

Let $\pm$ be a non-negative real number and let T be a positive integer. A game $(N;V)$ is a $(\pm;T)$-type game if there is a T-member ±-substitute partition $\{N[t] : t = 1,\ldots,T\}$ of N. The set $N[t]$ is interpreted as an approximate type. Players in the same element of a ±-substitute partition are ±-substitutes. When $\pm = 0$; they are exact substitutes.

pro‾les. Another notion that arises in the study of large games is that of the pro‾le of a player set, a vector listing the number of players of each type in a game. This notion is also employed in the de‾nition of a parameterized collection of games, but pro‾les are de‾ned relative to partitions of player sets into approximate types.

Let $\pm \geq 0$ be a non-negative real number, let $(N; V)$ be a game and let $fN[t] : t = 1; :::; Tg$ be a partition of N into $\pm$-substitutes. A pro‾le relative to $fN[t]g$ is a vector of non-negative integers $f \in Z^T_+$ and a subpro‾le s of a pro‾le f is a pro‾le satisfying the condition that $s \cdot f$. Given $S \subseteq N$ the pro‾le of S is a pro‾le, say $s \in Z^T_+$, where $s_t = jS \setminus N[t]j$: A pro‾le describes a group of players in terms of the numbers of players of each approximate type in the group. Let kfk denote the number of players in a group described by f, that is, $kfk = \sum f_t$.

<u>‾¡ e®ective B¡ bounded groups</u>: In all studies of approximate cores of large games, some conditions are required to limit gains to collective activities, such as boundedness of marginal contributions to coalitions, as in Wooders and Zame (1984,1989) or the less restrictive conditions of per capita boundedness and/or small group e®ectiveness, as in Wooders (1983,1994a,b), for example. Small groups are e®ective if all or almost all gains to collective activities can be realized by groups bounded in size of membership. The following notion formulates the idea of small e®ective groups in the context of parameterized collections of games.

Informally, groups of players containing no more than B members are ‾-e®ective if, by restricting coalitions to having fewer than B members, the loss to each player is no more than ‾: This is a form of small group e®ectiveness for arbitrary games. Let $(N; V)$ be a game. Let ‾ ≥ 0 be a given non-negative real number and let B be a given positive integer. For each group $S \subseteq N$; de‾ne a corresponding set $V(S; B) \subseteq R^N$ in the following way:

$$V(S; B) := \left[\left\{ \sum_k V(S^k) : \left\{ S^k \right\} \text{ is a partition of } S, \left\vert S^k \right\vert \cdot B \right\} \right].$$

The set $V(S; B)$ is the payo® set of the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S; B) \subseteq V(S)$ for any $S \subseteq N$ and, by construction, $V(S; B) = V(S)$ for $jSj \cdot B$. We might think of $c_q(V(S; B))$ as the payo® set to the coalition S when groups are restricted to have no more than B members and transfers are allowed between groups in the partition. If the game $(N; V)$ has q-comprehensive payo® sets then $c_q(V(S; B)) \subseteq V(S)$ for any $S \subseteq N$: The game $(N; V)$ with q-comprehensive payo® sets has ‾-e®ective B-bounded groups if for every group $S \subseteq N$

$$H_1[V(S); c_q(V(S; B))] \cdot \text{‾}.$$

When $^-$ = 0, 0-e®ective B-bounded groups are called strictly e®ective B-bounded groups.

<u>parameterized collections of games $G^q((\pm;T);(^-;B))$.</u> With the above de¯nitions in hand, we can now de¯ne parameterized collections of games.

Let T and B be positive integers and let q be a real number, $0 \cdot q \cdot 1$. Let $G^q((\pm;T);(^-;B))$ be the collection of all $(\pm;T)$-type games that have q-comprehensive payo® sets and $^-$-e®ective B-bounded groups.

Less formally, given non-negative real numbers $q;^-$ and $\pm;$ and positive integers T and B; a game (N;V) belongs to the class $G^q((\pm;T);(^-;B))$ if:

(a) the payo® sets satisfy q-comprehensiveness;

(b) there is a partition of the total player set into T sets where each element of the partition contains players who are $\pm$-substitutes for each other; and

(c) almost all gains to collective activities (with a maximum possible loss of $^-$ for each player) can be realized by partitions of the total player sets into groups containing fewer than B members.

Our results hold for all parameters $\pm$ and $^-$ that are su$\pm$ciently small, that is, $2(\pm + ^-) < m^{\pi}$; where m^{π} is a positive real number used in the de¯nition of the Hausdor® distance. (Since m^{π} can be chosen to be arbitrarily large, this requirement is nonrestrictive.)

<u>the equal-treatment property.</u> Let (N;V) be a $(\pm,T)$-type game, let x be a payo® vector and let $S \frac{1}{2} N$. We say that x is equal treatment on S if, for each $t = 1;::::;T$ and each $i;j \in N[t] \backslash S$; it holds that $x_i = x_j$. For $S = N$, we say that x has the equal-treatment property or is an equal treatment payo® vector.

The equal-treatment property of the core and approximate cores is treated in Kovalenkov and Wooders (1997b) and in a sequel to this paper. It's useful, however, in describing our results to have these de¯nitions in hand.

3   Non-emptiness of approximate cores of games.

<u>the core and the "-core.</u> Let (N;V) be a game. A payo® vector x is "-undominated if, for all $S \frac{1}{2} N$ and $y \in V(S)$; it is not the case that $y_S >> x_S + 1_S$". The payo® vector x is feasible if $x \in V(N)$. The "-core of a game (N;V) consists of all feasible and "-undominated payo® vectors. When " = 0, the "-core is the core.

3.1 The "-remainder core.

The concept of the "-remainder core is based on the idea that all requirements of the core should at least be satis¯ed for almost all players with the remainder of players representing a small fraction of \unemployed" or \underemployed" players. This approximate core notion can be viewed as a stepping stone to other notions of approximate cores. There are game-theoretic situations, however, in which the notion of the "-remainder core may naturally arise { for example, the demand games of Selten (1981).

the "-remainder core. Let $(N;V)$ be a game. A payo® vector x belongs to the "-remainder core if, for some group S ½ N, $\frac{|N|_i |S|}{|N|} \cdot$ " and x_S belongs to the core of the subgame $(S;V)$.

Note that the following theorem requires no restrictions on the degree of comprehensiveness { the usual notion of comprehensiveness su±ces.

Theorem 1. Non-emptiness of the "-remainder core. Let T and B be positive integers. For any " $> 0;$ there exists an integer $\acute{1}(";T;B)$ such that if
(a) $(N;V) 2 G^q((0;T);(0;B))$ and
(b) $|N| \, \dot{\,} \, \acute{1}(";T;B)$
then the "-remainder core of $(N;V)$ is non-empty.

While the assumptions of Theorem 1 are strong { a ¯xed number T of exact player types and strictly e®ective groups of size less than or equal to B; they provide a strong conclusion. The theorem states that for any " > 0 there exists a lower bound $\acute{1}(";T;B)$ on the number of the players such that all games satisfying the assumptions with more than $\acute{1}(";T;B)$ players have non-empty "-remainder cores. Since the bound depends only on "; T; and B, the bound is uniform across all the games characterized by the parameters; there is no restriction to replica games. Our result extends the result of Kaneko and Wooders (1982) from replication sequences to arbitrary large games. As in Kaneko and Wooders (1982) the result is independent of the characteristic function of the games; the same bound holds for all games in the collection parameterized by T and B.

To illustrate the application of Theorem 1, consider the collection of all games with at most two types of players and with two-person e®ective coalitions { marriage games, buyer-seller games, coalition production economies where two and only two workers are required for a productive coalition, and so on. Note that this collection of games cannot be described by a pregame { the collection is too large and cannot be accomodated by one space of player types and one worth function. Our result shows that given " $> 0;$ provided the number of players in the game is su±ciently large; any game in the collection has a nonempty "-remainder core. In Appendix 2 we provide another example illustrating application of the result.

3.2 Some preliminary intuition.

To give some intuition into Theorem 1 and the following Theorems, consider a particular game were all players are identical and only two-player coalitions are effective. It is immediate that the core of the game is nonempty if there is an even number of players. In fact, any even replication of a given game has a nonempty core. This doesn't depend on the payoff sets of the games; instead, it reflects the fact that any set with an even number of players can be partitioned into two-person "optimal" coalitions. When the total number of players is odd, one player can be assigned his individually rational payoff; thus, given $\varepsilon > 0$, for all sufficiently large total player sets, the ε-remainder core is nonempty.[12]

For sequences of games with a fixed distribution of player types and effective coalition sizes bounded by a positive integer B, for any game in the sequence there is a replication of the player set for which the core is non-empty — there are no "left-overs"; this follows from the fact that "minimal balancing weights" are rational numbers, an observation made by Shapley (1967) and applied in the study of large games in Wooders (1979,1980,1983).[13] Formally, there is an integer r_0 such that in the r_0th replicated game there are more than B players of each type. Since minimal balancing weights are rational numbers, there is an integer r_1 that clears all denominators of all minimal balancing weights for the r_0th replicated game. Effectively, balanced collections of coalitions for the r_0th replicated game become partitions in the $r_1 r_0$th replicated game. Thus, payoff vectors for the balanced cover of the r_0th game, replicated r_1 times, are feasible payoff vectors for the $r_1 r_0$th replicated game. Since no new improving opportunities arise when the r_0th game is replicated (from boundedness of effective group sizes) replications of payoff vectors in the core of the balanced cover of the r_0th game cannot be improved upon in the $r_1 r_0$th replicated game. Moreover, for any positive integer ℓ, payoffs in the core of the balanced cover of the r_0th game, replicated ℓr_1 times are in the cores of the $\ell r_1 r_0$th replicated game (Wooders (1979,1983), Kaneko and Wooders (1982)).

With effective group sizes bounded by B and a finite number of types of players, large player sets can be partitioned so that most players are in optimal coalitions. The argument above is for replication sequences. Demonstrating that this holds in

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[12]This intuition, for the case of identical players, already appears in Pauly (1970) and Shubik (1971). (Pauly's assumption in the two-type case, relating marginal contributions of players of different types to the worths of coalitions, are quite restrictive and in fact imply that there is no loss in efficiency in mixing both types. A paper demonstrating this result is available on request from Myrna Wooders.)

[13]The balanced cover of a game without side payments is another game with the same payoff sets for proper subsets of players and with the payoff set to the total player set enlarged sufficiently so that Scarf's (1967) Nonemptiness Theorem for the core of a balanced game can be applied. Roughly, balancedness of a game requires that there is no possible gain to allowing fractional coalitions, where the fractions can be assumed to be rational. The set of minimal balancing weights is just the smallest set of fractional coalitions that need be considered for Scarf's Theorem to apply.

general is the purpose of our fundamental proposition, the balancing e®ect of large numbers, provided in Section 5.

With the balancing e®ect of large numbers in hand, Theorem 1 is immediate. Theorems 2 and 3 follow by approximation arguments. The games considered approximate the class of games treated in Theorem 1.

3.3 The ε_1-remainder ε_2-core.

For a less restrictive de¯nition of the approximate core we can treat a signi¯cantly more general class of games than those of Theorem 1, in particular, we can allow approximate types (± > 0) and almost e®ective groups (¯ > 0). For example, the class of models covered by our next Theorem includes replica models of economies with private goods as in Debreu and Scarf (1963) and models of local public good economies satisfying per capita boundedness, as in Wooders (1988).

the ε_1-remainder ε_2-core. Let $(N;V)$ be a game. A payo® vector x belongs to the ε_1-remainder ε_2-core if there for some group $S \frac{1}{2} N$, $\frac{|N| - |S|}{|N|} \cdot \varepsilon_1$ and x_S belongs to the ε_2-core of the subgame $(S;V)$.

The following result extends the nonemptiness results of Wooders (1980,1983,1992), Shubik and Wooders (1983), and Wooders and Zame (1984,1989) from pregames to parameterized collections of games. For the same values of the parameters T and B the bound on the sizes of games in the following theorem can be chosen to equal the bound in the preceding theorem. Note that there are no restrictions on the value of q { strong comprehensiveness is not required.

Theorem 2. Non-emptiness of the ε-remainder $(± + ¯)$-core. Let T and B be positive integers. For any $\varepsilon > 0$ there exists an integer $´_1(\varepsilon; T; B)$ such that if
   (a) $(N;V) 2 G^q((±;T);(¯;B))$ and
   (b) $|N| ¸ ´_1(\varepsilon; T; B)$
then the ε-remainder $(± + ¯)$-core of $(N;V)$ is non-empty.

To describe the intuition behind Theorem 2, we begin with the same sort of example as discussed above where only two-player coalitions are e®ective. Suppose now that only two-player coalitions are e®ective but that the players are not identical, they are only all ±-substitutes for each other. Suppose that the total number of players is even. Consider a feasible payo® vector x that assigns the largest possible amount of payo® to each player, subject to the constraint that x has the equal-treatment property. Since all players are ±-substitutes, no two players can improve on x by more than ± for each player. Thus, x is in the ±-core. If there is an odd number of players and if $\varepsilon > \frac{1}{|N|}$ then the ε-remainder ±-core is nonempty; one player can be assigned his individually rational payo®.

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Now let us suppose that two-person coalitions are only ¯-e®ective. When jNj is even, selecting x as in the preceding paragraph will lead to a payo® vector in the (± + ¯)-core. When jNj is odd, the "-remainder (± + ¯)-core is nonempty for any " satisfying " > $\frac{1}{jNj}$; again, one player can be assigned his individually rational payo®. In e®ect, a parameterized collection of games $G^q((\pm; T); (\bar{\ }; B))$, by its de¯nition, can be approximated by the collection $G^q((0; T); (0; B))$:

Observe that by de¯nition the "-remainder 0-core coincides with the "-remainder core. Therefore, Theorem 2 is a strict generalization of Theorem 1 (Theorem 1 is a subcase for $\pm = \bar{\ } = 0$). But both Theorem 1 and Theorem 2 are based on the idea that some small proportion of the players can be ignored. An example in Appendix 2 illustrates this point.

3.4   The "-core.

Our third Theorem provides conditions for the non-emptiness of the "-core of large games. The proof is based on the idea of compensating the \remainder" players from the previous theorems, as in Wooders (1980,1983) and a number of subsequent papers. This compensation is possible under q-comprehensiveness (with q > 0) and one more condition, typically called per capita boundedness.

per capita boundedness. Let C be a positive real number. A game (N; V) has a per capita payo® bound of C if, for all coalitions S ½ N,

$$\sum_{a \in S} x_a \cdot C \, jSj \text{ for any } x \in V(S).$$

Theorem 3. Non-emptiness of the $(" + \pm + \bar{\ })$-core. Let T and B be positive integers. Let C and q be positive real numbers. Then for each " > 0 there exists an integer $´_2("; T; B; C; q)$ such that if:
  (a) $(N; V) \in G^q((\pm; T); (\bar{\ }; B))$,
  (b) (N; V) has per capita payo® bound C, and
  (c) $jNj \, ¸ \, ´_2("; T; B; C; q)$
then the $(" + \pm + \bar{\ })$-core of (N; V) is non-empty.

To describe the intuition behind Theorem 3, return again to the situation where all players are identical, only two-player coalitions are e®ective and jNj is odd. Suppose also that this is a game with side payments and any two-person coalition can earn 1; thus, the payo® vector $(\frac{1}{2}; ....; \frac{1}{2}; 0)$ is feasible. Then for any " satisfying $"jNj > \frac{1}{2}$ the payo® vector $(\frac{1}{2} ¡ "; ....; \frac{1}{2} ¡ "; y)$ is in the "-core where $y = (jNj ¡ 1)"$. Informally, we can make transfers of " from each of $jNj ¡ 1$ players in optimal coalitions to the reminder player so that every player has a payo® of at least $(\frac{1}{2} ¡ ")$. It follows that no group of players can signi¯cantly improve (by more than " in this case) upon such a payo®.

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The feature that per capita payo®s are bounded puts an upper bound on the required transfer to the remainder players. In games without side payments, the feature that q is greater than zero means that a small amount of payo® can be transferred from players in optimal coalitions to remainder players (but not necessarily at a one-to-one rate) until the remainder players are as well o® as those players in optimal coalitions, with the consequence that no group of players can signi¯cantly improve upon the resulting payo® vector.

Now return to the situation where two-player coalitions are ¯-e®ective and all players are ±-substitutes for each other. As we've observed, if jNj is odd, the (± + ¯)-core may be empty. When per capita payo®s are bounded and there is some means of making transfers (q > 0) then, for su±ciently large jNj; it's possible to begin with some payo® vector in the "-remainder (± + ¯)-core, and make transfers to the remainder players so that, no coalition could improve on the resulting payo® by " + ¯ + ± for each player.

We note that the assumption of per capita boundedness is actually stronger than required. For our proofs (as the proof of Wooders (1983)) we may use only the fact that the set of equal-treatment payo® vectors satisfy the per capita boundedness condition.

Indispensability of the conditions of Theorem 3 is demonstrated by the examples in Subsection 3.6 and in Appendix 2. The following Corollary shows that Theorem 3 can be applied to obtain non-emptiness of approximate cores of games that are \close" to q-comprehensiveness games (with q > 0).[14] The proof of this result is left to the reader.

Corollary. Non-emptiness with near q-comprehensiveness. Let (N; W) be a game. Suppose that for some q > 0; " > 0; and ° > 0 there exist a game (N; V) 2 G^q((±; T); (¯; B)) such that:
   (a) (N; V) has per capita payo® bound C,
   (b) jNj ¸ ´$_2$("; T; B; C; q) and
   (c) H$_1$ [W (S); V (S)] · $\frac{°}{2}$ for all S ½ N:
Then the (" + ± + ¯ + °)-core of (N; W) is non-empty.

In Wooders (1983), to obtain nonemptiness of approximate cores, q-comprehensiveness is not required. To obtain her equal-treatment result, however, as noted previously Wooders employs approximating games satisfying strong comprehensiveness. A similar idea is at work in our Corollary. If there is another game (N; V) with payo® sets close { within $\frac{°}{2}$ { to those of (N; W) and (N; V) satis¯es the conditions of Theorem 3, it is not necessary that the game (N; W) itself satis¯es those conditions. In these circumstances, we can ¯nd a payo® vector that is in the (" + ± + ¯)-core of (N; V)

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[14]Any comprehensive payo® set can be approximated arbitrarily closely by a q-comprehensive payo® set, for q small (Wooders 1983, Appendix).

and adjust this payoff vector so that it's in the ($\varepsilon + \pm + ^- + °$)-core of $(N; W)$. In essence, this is similar to the sort of approximation technique used for our Theorems 2 and 3.

3.5 A simple example of a production economy.

The following example illustrates the application of our results to a familiar sort of situation { one with firms and workers. The example also illustrates the application of our results to situations where there is a compact metric space of player types, essentially a special case.

> Example 1. Suppose a pregame has two sorts of players, firms and workers.[15] The set of possible types of workers is given by the points in the interval $[0; 1)$ and the set of possible types of firms is given by the points in the interval $[1; 2]$: To derive a game from the information given above, let N be any finite player set and let $\gg$ be an attribute function, that is, a function from N into $[0; 2]$. In interpretation, if $\gg(i) \in [0; 1)$ then i is a worker and if $\gg(i) \in [1; 2]$ then i is a firm. Firms can profitably hire up to three workers and the payoff to a firm i and a set of workers $W(i) \subseteq N$, containing no more than 3 members, is given by $v(\{i\} \cup W(i)) = \gg(i) + \sum_{j \in W(i)} \gg(j)$: Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any coalition $S \subseteq N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into coalitions containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game.
>
> We leave it to the reader to verify that for any positive integer m every game derived from the pregame is a member of the class $G^1((\frac{1}{m}; 2m); (0; 4))$ and has a per capita bound of 2. Theorem 3 states that given $\varepsilon > 0$; if $|N| \geq \gamma_2(\varepsilon; 2m; 4; 2; 1)$ then the game $(N; v)$ has a non-empty ($\varepsilon + \frac{1}{m}$)-core. In fact, Theorem 3 states this conclusion for an arbitrary game $(N; V)$ described by the same parameter values, $T = 2m; B = 4; \pm + ^- = \frac{1}{m}; C = 2$ and $q = 1$:

3.6 Per capita boundedness and small group effectiveness.

Our final result, Theorem 3, requires both per capita boundedness and small group effectiveness. As noted previously, in the context of pregames with side payments, when arbitrarily small percentages of players of any particular type is ruled out, then these two conditions are equivalent. But in important economic contexts, neither condition implies the other. The next example illustrates a voting games satisfying

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[15]We refer the reader to Wooders and Zame (1984) or Wooders (1992) for a definition of a pregame (with side payments) with a compact metric space of player types. For games without side payments, see Kaneko and Wooders (1996) or Wooders (1991), for example.

per capita boundedness. There is only one player type in each game so the \thickness" condition of Wooders's (1994b) equivalence result is satis‾ed. But small group e®ectiveness does not hold and Theorem 3 does not apply.

Example 2. Voting games. Consider a sequence of games $(N^m; v^m)^1_{m=1}$ with side payments and where the mth game has 3m players. Suppose that there are widespread positive externalities so that in the mth game, any coalition S consisting of at least 2m players can get up to 2m units of payo® to divide among its members, that is, $v^m(S) = 2m$. Assume that if $jSj < 2m$; then $v^m(S) = 0$.

We can think of the games as a sequence of voting games where a winning coalition must contain $\frac{2}{3}$ of the population, for example, impeachment of a President of the United States or rati‾cation of a treaty in some parliaments.

Observe that each game in the sequence has one exact player type and a per capita bound of 1. That is, $q = 1; T = 1; C = 1;$ and $\pm = 0$: However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m:

To see that the $\frac{1}{7}$-core is empty, observe that for any feasible payo® vector there are m players that are assigned, in total, no more than $\frac{2m}{3m}m = \frac{2}{3}m$: There are another m players that get in total no more than $\frac{2m}{2m}m = m$: These 2m players can form a coalition and receive 2m in total. This coalition can improve upon the given payo® vector for each of its members by $\frac{1}{6}$; since $(2m ¡ \frac{5}{3}m)\frac{1}{2m} = \frac{1}{6}$:

The following example, of matching games with widespread positive externalities, illustrates economic situations where, because of small group e®ectiveness, "-remainder cores are nonempty for positive ", but per capita boundedness does not hold and Theorem 3 does not apply.

Example 3. A matching game with widespread positive externalities.[16] The economic situation we've in mind is one where any two players can carry out some job but their reward from the job depends on the size of the economy in which they live. (It would be easy to modify the example to become a two or many-sided matching game.) Consider a sequence of games with side payments $(N^m; v^m)^1_{m=1}$ where the mth game has 2m + 1 players: Assume that any player alone can get only 0 units or less, that is $v^m(\text{fig}) = 0$ for all $i \in N$. Also assume that any two-player coalition can get up to 2m units of payo® to divide; $v^m(S) = 2m$ if $jSj = 2$. An arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than two players.

The games $(N^m; v^m)^1_{m=1}$ are members of the collection of games with one exact player type and strictly e®ective small groups of two. That is, $q = 1; T =$

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[16]A similar example in Wooders and Zame (1984).

1; B = 2; and $\pm = \bar{} = 0$: Given $" > 0$ the $"$-remainder core of the game $(N^m; v^m)$ is nonempty for all m satisfying $\frac{1}{2m+1} < "$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m; the remainder player cannot be compensated by the sum of taxes on the players in two-person coalitions:

To see that the $\frac{1}{7}$-core is empty, observe that for any feasible payo® vector there is a player whose payo® is no more than $\frac{2m^2}{2m+1}$: There is another player whose payo® must be no more than $\frac{2m^2}{2m} = m$: These two players may form a coalition and realize 2m: Thus they gain $m \, ¡ \, \frac{2m^2}{2m+1} = \frac{m}{2m+1} \, , \, \frac{m}{3m} = \frac{1}{3}$: Obviously, together this two-player coalition can improve upon the given payo® by $\frac{1}{6}$ for each member of the coalition:

3.7 Remarks.

Remark 1. q-comprehensiveness or convexity? It is possible to obtain a result similar to Theorem 3 using convexity of payo® sets and \thickness" instead of q-comprehensiveness (see Kovalenkov and Wooders 1997a). Strong comprehensiveness, however, can be naturally satis¯ed by games derived from economies. Moreover, \1-strongly comprehensive games" are games with side payments, so we can incorporate this important special case. Furthermore, in models of economies with local public goods or with clubs, convexity may be di±cult to satisfy. Although examples show that none of the assumptions can be omitted, our Corollary relaxes q-comprehensiveness.

Remark 2. Explicit bounds. It may be possible to compute the bounds on the size of the total player sets given in Theorems 1, 2, and 3 in terms of the parameters describing the games. A simple bound is obtained in Kovalenkov and Wooders (1997b), although under somewhat di®erent assumptions. Also, the proofs of that paper, relative to those of this paper, are quite complex.

Remark 3. Absolute or relative sizes? It is possible to obtain similar results with bounds on relative sizes of e®ective coalitions. In a ¯nite game with a given number of players, assumptions on absolute sizes and on relative sizes of e®ective coalitions are equivalent. We have chosen to develop our results using bounds on absolute sizes of near-e®ective coalitions since this seems to re°ect typical economic and social situations. Examples include: marriage and matching models (see Kelso and Crawford (1982) and Roth and Sotomayor (1990)); models of economies with shared goods and crowding (see Conley and Wooders (1998) for a survey); and private goods exchange economies (see Mas-Colell (1979) and Kaneko and Wooders (1989) for example). In fact, assumptions on proportions of economic agents typically occur only when there is a continuum of players, cf. Ostroy (1984).

Remark 4. Limiting gains to coalition formation. In the pregame framework several different conditions limiting returns to coalition formation have been used. For situations with a fixed distribution of a finite number of player types, Wooders (1980,1983) and Shubik and Wooders (1983) require per capita boundedness. To treat compact metric spaces of player types, Wooders and Zame (1984,1989) require boundedness of marginal contributions to coalitions while Wooders (1992,1994a,b) requires the less restrictive condition of small group effectiveness. As noted in the introduction, in the context of games derived from pregames, small group effectiveness and per capita boundedness are equivalent. In Subsection 3.5, we have shown that in the broader framework of parameterized collections of games both $^-$-effective B-bounded groups and per capita boundedness are required.

4   Economies with clubs.

We define admissible club structures in terms of natural properties and take as given the set of all admissible club structures for each coalition of agents. Generalizing Mas-Colell's (1980) notion of public projects to club activities, there is no necessary linear structure on the set of club activities. Indeed, our results could be obtained even without any linear structure on the space of private commodities. We remark that it would be possible to separate crowding types of agents (those observable characteristics that affect the utilities of others, or, in other words, their external characteristics) from taste types, as in Conley and Wooders (1996,1997), and have agents' roles as club members depend on their crowding types. In these papers, however, the separation of crowding type and taste type has an important role; the authors show that prices for public goods { or club membership prices { need only depend on observable characteristics of agents and not on their preferences. The current paper treats only the core so the separation of taste and crowding type would have no essential role and therefore is not made.

agents. There are T \types" of agents. Let $m = (m_1; ::::; m_T)$ be a given profile, called the population profile. The set of agents is given by

$$N_m = f(t; q) : q = 1; ::::; m_t \text{ and } t = 1; ::::; T \ g;$$

and $(t; q)$ is called the qth agent of type t. It will later be required that all agents of the same type may play the same role in club structures. For example, in a traditional marriage model, all females could have the role of \wife". Define $N_m[t] := f(t; q) : q = 1; ::::; m_t \ g$. For our first Proposition members of $N_m[t]$ will be exact substitutes for each other and for our next two Propositions, approximate substitutes.

commodities. The economy has L private goods. A vector of private goods is denoted by $y = (y_1; ::::; y_`; ::::; y_L) \ 2 \ R_+^L$.

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<u>clubs</u>. A club is a subset of agents. For each $S \in N_m$, a club structure of S, denoted by $\mathcal{S}$, is a set of clubs whose union coincides with S: The non-empty set of admissible club structures for S is denoted by $\mathcal{C}(S)$. These sets are required to satisfy the following two properties:

1. If S and S^0 are disjoint subsets of agents and $\mathcal{S}$ and $\mathcal{S}^0$ are club structures of S and S^0 respectively, then $\{C : C \in \mathcal{S} \cup \mathcal{S}^0\}$ is a club structure of $S \cup S^0$ (unions of admissible club structures of disjoint coalitions are club structure of the unions of the coalitions).

2. Let S and S^0 be subsets of agents with the same pro¯les, let $\mathcal{S}$ be a club structure of S and let $'$ be a type-preserving 1-1 mapping from S onto S^0 (that is, if $(t; q) \in S$ then $'((t; q)) = (t; q^0)$ for some $q^0 = 1; ::::; m_t$). Then

$$\mathcal{S}^0 = \{C \in S^0 : '^{i\,1}(C) \in \mathcal{S}\}$$

is a club structure of S^0 (admissible club structures depend only on pro¯les, that is, all agents of the same type have the same roles in clubs).

Note that it follows from the de¯nition of the admissible club structures that the partition of any set S into singletons is always an admissible club structure for S. The ¯rst assumption is necessary to ensure that the game derived from the economy is superadditive. It corresponds to economic situations where one option open to a group is to form smaller groups. The second assumption corresponds to the idea that the opportunities open to a group depend on the pro¯le of the group.

<u>club activities</u>. For each club C there is a set of club activities $A(C)$: An element $\circledR$ of $A(C)$ requires input $x(C; \circledR) \in R^L$ of private goods. For any two clubs C and C^0 with the same pro¯le we require that if $\circledR \in A(C)$, then $\circledR \in A(C^0)$ and $x(C; \circledR) = x(C^0; \circledR)$: For 1-agent clubs $\{(t; q)\}$, we assume that there is an activity $\circledR_0$ with $x(\{(t; q)\}; \circledR_0) = 0$, that is, there is an activity requiring no use of inputs.

<u>preferences and endowments</u>. Only private goods are endowed. Let $!^{tq} \in R^L_+$ be the endowment of the $(t; q)^{th}$ participant of private goods.

Given $S \in N_m$, $(t; q) \in S$, and a club structure $\mathcal{S}$ of S, the consumption set of the $(t; q)^{th}$ agent (relative to $\mathcal{S}$) is given by

$$\copyright^{tq}(\mathcal{S}) := X^{tq}(\mathcal{S}) \pounds \prod_{C \in \mathcal{S}} A(C);$$

where $X^{tq}(\mathcal{S}) \in R^L$ is the private goods consumption set relative to $\mathcal{S}$, assumed to be closed. Thus, the entire consumption set of the $(t; q)^{th}$ agent is given by

$$\copyright^{tq} := \bigcup_{S \in N_m : (t;q) \in S} \bigcup_{\mathcal{S} \in \mathcal{C}(S)} \copyright^{tq}(\mathcal{S}):$$

We assume that the $(t, q)^{th}$ agent can subsist in isolation. That is

$$(! ^{tq}, ®_0) \ 2 \ ©^{tq} \ (f(t, q)g) :$$

It is also assumed that for each (t, q), each $S \ ½ \ N_m$, $(t, q) \ 2 \ S$, and each club structure S of S, the preferences of the $(t, q)^{th}$ agent are represented by a continuous utility function $u^{tq}(¢, S)$ de¯ned on $©^{tq}(S)$.

states of the economy. Let S be a non-empty subset of N_m and let S be a club structure of S. A feasible state of the economy S relative to S, or simply a state for S, is a pair $(y^S, ®^S)$ where:

(a) $y^S = fy^{tq}g_{(t,q)2S}$ with $y^{tq} \ 2 \ X^{tq}(S)$ for $(t, q) \ 2 \ S$;

(b) $®^S = f®^Cg_{C2S}$ with $®^C \ 2 \ A(C)$ for $C \ 2 \ S$; and

(c) the allocation of private goods is feasible, that is,
$$\sum_{C2S} x(C, ®^C) + \sum_{(t,q)2S} y^{tq} = \sum_{(t,q)2S} ! ^{tq}:$$

feasible payo®s vectors. A payo® vector $U = (û^{tq})_{(t,q)2N_m}$ is feasible for a coalition S if $ū^{tq} = 0$ for all $(t, q) \ 2 \ N_m nS$ and there is club structure S of S and a feasible state of the economy for S relative to S, $(y^S, ®^S)$, such that $ū^{tq} = u^{tq}(y^{tq}, ®^S, S)$ for each $(t, q) \ 2 \ S$.

the game induced by the economy. For each coalition $S \ ½ \ N_m$, de¯ne

$$V(S) = f(b^{tq})_{(t,q)2N_m} : \text{ there is a payo® vector } (û^{tq})_{(t,q)2N_m}$$
$$\text{that is feasible for } S \text{ and } b^{tq} \ ¢ \ ū^{tq} \text{ for all } (t, q) \ 2 \ Sg:$$

It is immediate that the player set N_m and function V determine a game (N_m, V) with comprehensive payo® sets.

"-domination. Let N_m be a club structure of the total agent set N_m and let $(y^{N_m}, ®^{N_m})$ be a feasible state of the economy N_m relative to N_m. A coalition S can "-dominate the state $(y^{N_m}, ®^{N_m})$ if there is a club structure $S = fS_1, ::::, S_Kg$ of S and a feasible state $(y^{0S}, ®^{0S})$ for the economy S such that for all consumers $(t, q) \ 2 \ S$ it holds that

$$u^{tq}(y^{0tq}, ®^{0S}, S) > u^{tq}(y^{tq}, ®^S, N_m) + ":$$

the core of the economy and "-cores. The state $(y^{N_m}, ®^{N_m})$ is in the core of the economy if it cannot be improved upon by any coalition S. It is clear that if $(y^{N_m}, ®^{N_m})$ is a state in a core of the economy then the utility vector induced by that state is in the core of the induced game. Similarly, if $(û^{tq})_{(t,q)2N_m}$ is in a core of the game then there is a state in the core of the economy $(y^{N_m}, ®^{N_m})$ such that the utility vector induced by that state is $(û^{tq})_{(t,q)2N_m}$. Notions of the "-remainder core, the "-remainder "-core, and the "-core of the economy are de¯ned in the obvious way.

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4.1 Non-emptiness of approximate cores.

To obtain our results we require few restrictions on the economy. Our first Proposition requires exact player types and strictly effective small groups.

(A.0) For each t and all $q, q' \in \{1, \ldots, m_t\}$, $u^{tq}(\cent) = u^{tq'}(\cent)$ and $!^{tq} = !^{tq'}$: In addition, in the game induced by the economy the players (t, q) and (t, q') are exact substitutes. (All agents of the same type are identical in terms of their endowments, preferences and crowding types { their effects on others.)

(A.1) There is a bound B such that for any population profile m; any coalition $S \subseteq N_m$; and any club structure S of S, if $U = (\mathring{u}^{tq} : (t, q) \in S)$ is a feasible payoff vector for the club structure S then there is a partition of S into coalitions, say $\{S^1, \ldots, S^K\}$ and club structures of these coalitions, $\{S^1, \ldots, S^K\}$ such that for each k $\bar{S}^{k-} \cdot B$ and $U^k := (\bar{u}^{tq} : (t, q) \in S^k)$ is a feasible payoff vector for S^k:

Our approach requires that the set of individually rational and feasible outcomes is compact. It is possible to introduce conditions on the primitives of the economy as, for example, Debreu's (1962) condition of positive semi-independence, but for the purposes of this application, we will simply assume compactness.

(A.2) For each subset of agents $S \subseteq N_m$ the set $V(S) \setminus R_+^N$ is compact.

The following result is an immediate application of Theorem 1.

Proposition 1. Non-emptiness of the "-remainder core. Let T and B be positive integers. Assume (A.0)-(A.2) hold. Given $" > 0$; there exists an integer $\acute{}_1(", T, B)$ such that if m, the profile of the economy, satisfies the property that $\text{kmk} \cdot \acute{}_1(", T, B)$ then the "-remainder core of the economy is non-empty.

Proposition 1 is most natural if there is only one private good or if private goods are indivisible so that all gains from trade in private goods can be realized by trade within coalitions of bounded sizes. If we require only non-emptiness of the "-remainder "-core we can weaken the restrictions on the economy { players of the same type need only be approximate substitutes and small groups need only be nearly effective. For brevity, these assumptions will not be made on the primitives of the economy. Thus, instead of (A.0) and (A.1), for the following two Propositions, we will assume (A.0') and (A.1'):

(A.0') For some $\pm \cdot 0$ the players in the set $N_m[t] = \{(t, q) : q = 1, \ldots, m_t\}$ are $\pm$-substitutes for each other in the game induced by the economy.

(A.1') There is an $^- \cdot 0$ and an integer B so that the game derived from the economy has $^-$-effective B-bounded groups.

Then the following result, for the $"_1$-reminder $"_2$-core, follows from Theorem 2.

Proposition 2. Non-emptiness of the "-remainder $(\pm + \bar{\ })$-core. Let T and B be positive integers. Assume (A.0^0), (A.1^0) and (A.2) hold. Given $" > 0$; there exists an integer $\acute{}_1(";T;B)$ such that if m, the pro$\bar{}$le of the economy, satis$\bar{}$es the property that kmk $_\circ$ $\acute{}_1(";T;B)$ then the "-remainder $(\pm + \bar{\ })$-core of the economy is non-empty.

For our next result, we require that each agent always owns some commodity which other agents value. Speci$\bar{}$cally, we assume that the Lth commodity is a \quasi-money" with which everyone is endowed and for which everyone has a separable preference. In the following, let $y^{tq}_{i~L;}$ and $X^{tq}_{i~L}(S)$ denote the restriction of y^{tq} and $X^{tq}(S)$ respectively to their $\bar{}$rst $(L_i~1)$ coordinates. We also assume that, for some real number $q^{\pi}~2~(0;1]$, the marginal utility of the Lth commodity, for all su$\pm$ciently large amounts of the commodity, is greater than or equal to q^{π}:

(A.3) (q^{π}-comprehensiveness): For good L and all participants $(t;q)~2~N_m$ there is a positive real number $^\circ$ such that $!^{tq}_L~_\circ~^\circ > 0$ (everybody is endowed with the Lth good). Moreover, for any state of the economy $(y^S;®^S)$ we have :

(a) $X^{tq}(S) = X^{tq}_{i~L}(S) \pounds R_+$ (the consumption set is separable and the projection of the Lth coordinate is R_+),

(b) $u^{tq}(y^{tq};®^S;S) = u^{tq}_{i~L}(y^{tq}_{i~L};®^S;S) + u^{tq}_L(y^{tq}_L;®^S;S)$ for some functions $u^{tq}_{i~L}(\cent;\cent)$ and $u^{tq}_L(\cent;\cent)$ (utility is separable),

(c) for a real number $q^{\pi};~0 < q^{\pi} \cdot 1;$ for all players $(t;q)$ the marginal utility of the $(t;q)^{th}$player for the Lth good on the range $(\frac{\circ}{2};1)$ is between q^{π} and 1.

(A.4) (per capita boundedness): There is a constant C such that the condition of per capita boundedness is satis$\bar{}$ed by the games derived from the economies.

Assumption (A.3) ensures that the remainder players can be \paid o®," (at the rate q^{π}) so that they cannot pro$\bar{}$tably join improving coalitions.[17] Alternatively, it could simply be assumed that utility functions are linear in one commodity. (That is, $u^{tq}(x^{tq};®^S;S) = u^{tq}_{i~L}(x^{tq}_{i~L};®^S;S) + x^{tq}_L;$) This implies q^{π}-comprehensiveness of the game derived from the economy. Then the next result follows from Theorem 3.

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[17]Such an assumption, in the literature on private goods economies with indivisibilities, goes back at least to Broome (1972). It was introduced in the club/local public good literature in several papers due to Wooders, for example, Wooders (1988). It is possible to weaken (A.3) to the sort of assumption introduced in Hammond, Kaneko and Wooders (1989), that the endowment is preferred to any bundle containing only indivisible (in our case, club) goods but, for our purposes here, we prefer assumptions that make the results transparent.

Proposition 3. Non-emptiness of the $(" + \pm + ^-)$-core. Let T and B be positive integers. Let C be a per capita bound and q^{π} be positive real numbers. Assume that $(A.0^0)$, $(A.1^0)$, $(A.2)$-$(A.4)$ hold. Given $" > 0$; there exists an integer $´_2(";T;B;C;q^{\pi};°)$ such that if m; the pro¯le of the economy, satis¯es kmk ¸ $´_2(";T;B;C;q^{\pi};°)$ then the $(" + \pm + ^-)$-core of the economy is non-empty.

4.2 Further applications.

The class of economies de¯ned above is very broad. The results can be applied to extend results already in the literature on economies with coalition structures, such as those with local public goods (called club economies by some authors), cf., Shubik and Wooders (1982).

For example, there are a number of papers showing core-equilibrium equivalence in ¯nite economies with local public goods and one private good and satisfying strict e®ectiveness of small groups, cf., Conley and Wooders (1995) and references therein. In these economies, from the results of Wooders (1983) and Shubik and Wooders (1983), existence of approximate equilibrium where an exceptional set of agents is ignored is immediate. (Just take the largest subgame having a non-empty core and consider the equilibria for that subeconomy; ignore the remainder of the consumers.) Our results allow the immediate extension of these results to results for all su±ciently large economies { no restriction to replication sequences is required.

5 Mathematical foundations: Partition-balanced pro¯les.

This section formalizes some key ideas about pro¯les that underlie the non-emptiness of approximate cores of large games. Throughout this section, let the number of types of players be ¯xed at T. Thus, every pro¯le f has T components and f 2 R^T. Our key de¯nitions follow.

B{partition-balanced pro¯les. A pro¯le f is B-partition-balanced if any game $(N;V)$ 2 $G^q((0;T);(0;B))$ where the pro¯le of N is f (that is, jN[i]j = f_i for any i = 1; :::; T) has a nonempty core.

replicas of a pro¯le. Given a pro¯le f and a positive integer r; the pro¯le rf is called the rth replica of f:

The Lemma below is a very important step. It states that, for any pro¯le f; there is a replica of that pro¯le that is B-partition-balanced. The smallest such replication number is called the depth of the pro¯le. Note that the depth of a pro¯le depends on the pro¯le.

Lemma 1. (Kaneko and Wooders, 1982, Theorem 3.2)[18] The balancing e®ect of replication. Let B be a positive integer and let f̄ be any pro¯le. Then there is an integer r(f̄; B); the depth of f̄, such that, for any positive integer k, the pro¯le kr(f̄; B)f̄ is B-partition-balanced.

We refer the reader to Kaneko and Wooders (1982) for a proof. This result partially extends the earlier result, discussed in Subsection 3.2, that for sequences of games with a ¯xed distribution of player types, when all gains to improvement can be realized by coalitions bounded in size, there is a replication of any game in the sequence with the property that the replicated game has a nonempty core (Wooders (1979, 1983)). The following concept of "-B-partition-balanced pro¯les completes our construction.

"-B-partition-balanced pro¯les. Given a positive integer B and a non-negative real number "; 0 · " · 1; a pro¯le f̄ is "-B-partition-balanced if there is a subpro¯le f̄⁰ of f̄ such that $\frac{kf^{0}k}{kfk}$ ¸ 1 ¡ " and f̄⁰ is B{partition-balanced.

The next result is key: given " > 0 and B; any su±ciently large pro¯le is "-B-partition-balanced. Note that this result is uniform across all large pro¯les.

Fundamental Proposition. The balancing e®ect of large numbers. Given a positive integer B and a positive real number "; 0 < " · 1; there is a positive integer k("; B) such that any pro¯le f̄ with kf̄k ¸ k("; B) is "-B-partition-balanced.

The idea of the proof: Lemma 1 provides a way to replicate a pro¯le that ensures "-B-partition-balancedness of the resulting replica. The manner of replication depends on the initially given pro¯le. Using Lemma 1, however, for any given " we can construct a number of \small" pro¯les that, when appropriately replicated, create B-partition-balanced replicas that \"-approximate" all su±ciently large pro¯les. The proof is presented in Appendix 1. ∎

6 Conclusions.

Except in certain idealized situations, cores of games are typically empty. This has the consequences that important classes of economies typically have empty cores and a competitive equilibrium does not exist. Examples include economies with indivisibilities and other nonconvexities, economies with public goods subject to crowding, and production economies with non-constant returns to scale. The standard justi¯cation for convexity, assumed in Arrow-Debreu-McKenzie models of exchange economies, is that the economies are \large," rendering nonconvexities negligible { the convexifying

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[18]For a recent discussion and an interesting application of this sort of result to dynamic matching processes, see Myerson (1991).

effect of large numbers. Similarly results on non-emptiness of approximate cores rely on large numbers of players and the balancing effect of large numbers. An important aspect of our results in this paper is that they are for arbitrary games and the bounds depend on the parameters describing the games; the compact metric space of player types assumed in previous work is a special case. Moreover, our approach allows both widespread externalities and uniform results.

It appears that the framework of parametrized collections of games and our approach will have a number of uses. In ongoing research this framework is used to demonstrate further market-like properties of arbitrary games[19]: approximate cores are nearly symmetric { treat similar players similarly; arbitrary games are approximately market games and; arbitrary games satisfy a \law of scarcity," dictating that an increase in the abundance of players of a given type does not increase the core payoff vectors to members of that type. In addition, some initial results have been obtained on convergence of cores and approximate cores. A particularly promising direction appears to be the application of ideas of lottery equilibrium in games of Garratt and Qin (1996) to parameterized collections of games. Another possible application is to games with asymmetric information, as in Allen (1994), for example, and Forges (1998).[20]

7 Appendix 1: Proofs.

A technical lemma is required. Denote by $\|\cdot\|$ the sum-metric in R^T, that is, for $x, y \in R^T$; $\|x - y\| := \sum_i |x_i - y_i|$. Let us consider the simplex in R^T_+: $\Delta_+ := \left\{ \zeta \in R^T_+ : \sum_{i=1}^T \zeta_i = 1 \right\}$: For any positive integer γ; let us define the following finite set in the simplex:

$$\Delta_\gamma := \left\{ x \in \Delta_+ : \gamma x \in Z^T_+ \right\}:$$

Finally, let us define $\gamma(\varepsilon) = \min \left\{ \gamma \in Z_+ : \gamma \geq \frac{T}{\varepsilon} \right\}$. Now we can state the lemma.

Lemma 2. For each $\varepsilon > 0$ and for any $f \in R^T_+$ there is a vector $g \in R^T_+$ satisfying $g \leq f$; $\|f\| - \|g\| = \|f - g\| \leq \varepsilon\|f\|$ and $\frac{g}{\|g\|} \in \Delta_{\gamma(\varepsilon)}$:

Proof of Lemma 2: Let us first prove that $(1 + \varepsilon)\Delta_+ \subseteq \Delta_{\gamma(\varepsilon)} + \varepsilon\Delta_+$: Consider any $a = (a_1; \ldots; a_T) \in (1 + \varepsilon)\Delta_+$: (That is $\sum_{i=1}^T a_i = 1 + \varepsilon$ and $a_i \geq 0$ for each $i = 1; \ldots; T$:) Let us define $I^k \in R^T$ such that $I^k_l = 1$ for $k = l$ and 0 otherwise. Notice that $I^k \in \Delta_{\gamma(\varepsilon)}$ for any k: If there exist j such that $a_j \geq 1$; then $(a - I^j) \in \varepsilon\Delta_+$ and thus

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[19]See Shapley and Shubik (1966,1969) for seminal results of this nature and Wooders (1994a,b) for more recent references to related results in the context of pregames and economies with clubs/local public goods.

[20]We are grateful to Francoise Forges for pointing out this possible application.

$a = I^j + (a_j \; I^j) \; 2 \; 4_{\cdot(")} + " \; 4_+$: If $a_j < 1$ for any j; then let us consider $a_j = \frac{I_j}{\cdot(")} + r_j$, where I_j is an integer, $I_j < \acute{}(")$; and $0 \cdot r_j < \frac{1}{\cdot(")}$: Then

$$a = (a_1; :::; a_T) \; 2 \; (\frac{I_1}{\cdot(")}; :::; \frac{I_T}{\cdot(")}) + \frac{T}{\cdot(")} 4_+ \; \frac{1}{2} \; 4_{\cdot(")} + \frac{T}{\cdot(")} 4_+ \; \frac{1}{2} \; 4_{\cdot(")} + " 4_+:$$

Now given a pro¯le f, observe that $\frac{f}{kfk} \; 2 \; 4_+ \; \frac{1}{2} \; (4_{\cdot(")} + " \; 4_+)\frac{1}{1+"}$: Therefore there exists $h \; 2 \; \frac{1}{1+"} 4_{\cdot(")}$ such that $\frac{f}{kfk} \; 2 \; (fhg + \frac{"}{1+"} 4_+)$: Now, de¯ne $g := h \, kfk$: Then $g \cdot f$ and, by construction, $\frac{g}{kgk} = (1 + ")h \; 2 \; 4_{\cdot(")}$: Moreover $\frac{kfk_i \, kgk}{kfk} = 1 \; \textrm{¡} \; \frac{1}{1+"} = \frac{"}{1+"} < "$: ∎

Proof of Fundamental Proposition: Given a positive integer $\acute{}$; we ¯rst de¯ne an integer that will play an important role in the proof. Arbitrarily select $x \; 2 \; 4_{\cdot}$ and de¯ne $y(x) := \acute{}x \; 2 \; Z_+^T$: Since $y(x)$ is a pro¯le, by Lemma 1 there is an integer $r(y(x); B)$ such that for any integer k the pro¯le $kr(y(x); B)y(x)$ is B-partition-balanced: There exists such an integer $r(y(x); B)$ for each $x \; 2 \; 4_{\cdot}$: Since $4_{\cdot}$ contains only a ¯nite number of points, there is a ¯nite integer $M(\acute{}; B)$ such that $\frac{M(\acute{};B)}{r(y(x);B)}$ is an integer for any $x \; 2 \; 4_{\cdot}$:

By Lemma 2, given $\frac{"}{2} > 0$; there exists a positive integer $\acute{}^0 := \acute{}(\frac{"}{2})$; such that for any $f \; 2 \; R_+^T$ there exists a vector $g \; 2 \; R_+^T$ satisfying

$$g \cdot f;$$
$$kfk \; \textrm{¡} \; kgk = kf \; \textrm{¡} \; gk \cdot \frac{"}{2} kfk \text{ and}$$
$$\frac{g}{kgk} \; 2 \; 4_{\cdot^0}:$$

Arbitrarily select $f \; 2 \; R_+^T$ and let $g \; 2 \; R_+^T$ be a vector satisfying the above conditions. De¯ne $y^\alpha := \acute{}^0 \frac{g}{kgk}$. Since $\frac{g}{kgk} \; 2 \; 4_{\cdot^0}$; it holds that $y^\alpha \; 2 \; Z_+^T$: Therefore y^α is a pro¯le. Moreover, by the choice of $M(\acute{}^0; B)$; the $kM(\acute{}^0; B)^{th}$-replica of the pro¯le y^α is B-partition-balanced for any integer k:

Observe that there is an integer k^0; possibly equal to zero, such that

$$k^0 M(\acute{}^0; B)y^\alpha \cdot g < (k^0 + 1)M(\acute{}^0; B)y^\alpha:$$

De¯ne

$$f^0 := k^0 M(\acute{}^0; B)y^\alpha = k^0 M(\acute{}^0; B) \acute{}^0 \frac{g}{kgk}:$$

Obviously, f^0 is a pro¯le ($f^0 \; 2 \; Z_+^T$) and $f^0 \cdot g \cdot f$: Suppose that $k^0 > 0$: Then

$$\frac{f^0}{kf^0k} = \frac{g}{kgk} \text{ and } kgk \; \textrm{¡} \; kf^0k = kg \; \textrm{¡} \; f^0k \cdot M(\acute{}^0; B) \acute{}^0:$$

Moreover, the pro¯le f^0 is B-partition-balanced since it is a replica of the pro¯le y^α:

Now, de¯ne $k(";B) := M(´^0;B)^{´0}\frac{2}{"}$: If $\|f\| \geq k(";B)$, then

$$k^0 > 0; \quad f^0 \cdot g \cdot f,$$
$$\|f - g\| \cdot \frac{"}{2}\|f\|; \text{ and}$$
$$\|g - f^0\| \cdot M(´^0;B)^{´0} \cdot \frac{"}{2}\|f\|:$$

Therefore $\|f\| - \|f^0\| = \|f - f^0\| \cdot "\|f\|$: Thus f^0 is a subpro¯le of f, $\frac{\|f^0\|}{\|f\|} \geq 1 - "$; and f^0 is B-partition-balanced. ∎

Proof of Theorem 1: Fix the number of types T and consider the bound $k(";B)$ from the Fundamental Proposition. Let $´_1(";B;T) := k(";B)$: Let $(N;V)$ be a game with $|N| \geq k(";B)$. Denote the pro¯le of N by f: By the Fundamental Proposition, f is "-B-partition-balanced. That is, there is a B-partition-balanced subpro¯le f^0 of f such that $\frac{\|f^0\|}{\|f\|} \geq 1 - "$: Now select some $S \frac{1}{2} N$ such that $|S[i]| = f_i^0$ for any $i = 1;:::;T$: Then $\frac{|N| - |S|}{|N|} \cdot "$ by choice of S and the subgame $(S;V)$ has a non-empty core. Thus the "-remainder core of $(N;V)$ is non-empty. ∎

Proof of Theorem 2: For any $S \frac{1}{2} N$ de¯ne $V^0(S) := \bigcap^T_{¿} \frac{1}{4}^{-1}(V(¿(S)))$; where the intersection is taken over all type-preserving permutations $¿$ of the player set N. Then $(N;V^0) 2 G^q((0;T);(¯;B))$. Moreover, from the de¯nition of $V^0(S)$ it follows that $V^0(S) \frac{1}{2} V(S)$: (Informally, taking the intersection over all type-preserving permutations makes all players of each approximate type no more productive than the least productive members of that type.) From the de¯nition of ±-substitutes, it follows that $H_1[V^0(S);V(S)] \cdot ±$ for any $S \frac{1}{2} N$.

Now for any $S \frac{1}{2} N$, de¯ne $V^q(S) := c_q(V^0(S;B))$. Then $(N;V^q) 2 G^q((0;T);(0;B))$. Moreover, $V^q(S) \frac{1}{2} V^0(S) \frac{1}{2} V(S)$ and $H_1[V^q(S);V(S)] \cdot H_1[V^q(S);V^0(S)] + H_1[V^0(S);V(S)] \cdot ¯ + ±$.

By Theorem 1, if $|N| \geq ´_1(";T;B)$ then the "-remainder core of the game $(N;V^q)$ is non-empty. That is, there exists $S \frac{1}{2} N$ such that $\frac{|N| - |S|}{|N|} \cdot "$ and such that $(S;V)$ has a non-empty core. Let x be a payo® vector in the core of the game $(S;V^q)$. Since $V^q(S) \frac{1}{2} V(S)$, the payo® vector x is feasible and $(¯ + ±)$-undominated for the game $(S;V)$. Thus, the "-remainder $(± + ¯)$-core of $(N;V)$ is non-empty. ∎

Proof of Theorem 3: As in the proof of Theorem 2 ¯rst construct the game $(N;V^q) 2 G^q((0;T);(0;B))$. As noted in the proof of Theorem 2, $V^q(S) \frac{1}{2} V(S)$ and $H_1[V^q(S);V(S)] \cdot ¯ + ±$ for any $S \frac{1}{2} N$. In addition, the game $(N;V^q)$ has a per capita bound of C. We required that $2(¯ + ±) < m^¤$. Assume ¯rst that $2" \cdot m^¤$: Thus $(" + ¯ + ±) < m^¤$.

Applying Theorem 1 for $"^0 := \frac{q}{BC}"$ to the game $(N;V^q)$ we found that for $|N| \geq ´_1("^0;B;T)$ there is some subset of players $S \frac{1}{2} N$ with $\frac{|N| - |S|}{|N|} \cdot "^0$ such that the game $(S;V^q)$ has a non-empty core. Let x be a payo® vector in the core of $(S;V^q)$.

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We now construct a payoff vector y 2 R^N for the game $(N; V)$: For a 2 S, de¯ne $y_a := x_a ¡ "$ and for a $\notin$ S, de¯ne $y_a := BC ¡ "$. Observe that y is in the "-core of the game $(S; V^q)$.

We next need to show that y 2 $V^q(N)$. Since $\frac{jNj ¡ jSj}{jNj} \cdot "^0 = \frac{q}{BC}"$, it holds that

$$q"jNj ¸ BC(jNj ¡ jSj):$$

Since $q \cdot 1$, it follows that

$$q"jSj ¸ (BC ¡ ")(jNj ¡ jSj):$$

Informally, this means that we can take $"$ away from each player in S, transfer this amount to the players in NnS at the rate q; and increase the payoff to each player in NnS to $BC ¡ "$: Therefore since x 2 $V^q(S)$; by superadditivity and by q-comprehensiveness of payoff sets it holds that y 2 $V^q(N)$.

We now prove that the payoff vector y is "-undominated in the game $(N; V^q)$. The strategy of the proof is to show that if y is "-dominated in the game $(N; V^q)$ then it can be "-dominated by some coalition (to be called) A ½ S: We thus obtain a contradiction. The proof proceeds through two steps.

The ¯rst step is to construct the coalition A. Suppose that y is "-dominated in the game $(N; V^q)$ by some coalition W: Speci¯cally, suppose there exists a payoff vector z such that

$$z \ 2 \ V^q(W) = c_q(V^q(W; B)) \ \text{and}$$

$$z_W >> y_W + \mathbb{1}_W ":$$

Since z 2 $V^q(W)$ there exists some partition $\{W^k\}$ of W; $\overline{W^{k¯}} \cdot B$ and some payoff vector z^0 2 $\prod_k V^q(W^k)$ such that z can be obtained from z^0 by making \transfers" at the rate q between agents in W: Let

$$A := \bigcup \{W^k : W^k ½ S\} \ \text{and let} \ A^L := \bigcup \{W^k : W^k nS \neq ;\};$$

that is, A consists of those members of subsets in $\{W^k\}$ that are contained in S and A^L consists of those members of subsets of $\{W^k\}$ that contain at least one player from NnS:

The second step is to show that the set A is non-empty and can "-dominate the payoff vector y. Since y is in the "-core of the subgame $(S; V^q)$ it is clear that the coalition W must contain at least one member of NnS; therefore the set A^L must be non-empty. Observe that for any W^k ½ A^L and x^0 2 $V^q(W^k)$; it holds that $\sum_{a2S^k} x_a^0 \cdot BC$: There exists, however, a 2 $W^k nS$ such that $z_a >> y_a + " = BC$: Thus, z can be feasible in $V^q(W)$ only by some transfers from the players in the set A to the players in the set A^L: This implies that the set A is non-empty. Moreover the

coalition A is not a net bene¯ciary of transfers needed to support the payo® vector z: This implies that there is a payo® vector z^{00} 2 $V^q(A)$ such that for all players a 2 A;

$$z_a^{00} \, \, z_a > y_a + \text{":}$$

Since A ½ S; this is a contradiction to the construction of y as a payo® vector in the "-core of the game $(S; V^q)$: We conclude that y is "-undominated in the game $(N; V^q)$.

Since the payo® vector y is "-undominated in the game $(N; V^q)$, for jNj , $´_1(\frac{q}{BC}\text{"}; B; T)$ the payo® vector y is in the "-core of the game $(N; V^q)$. This implies that y is feasible and (" + ¯ + ±)-undominated in the initial game $(N; V)$, providing that jNj , $´_1(\frac{q}{BC}\text{"}; B; T)$. Let $´_2(\text{"}; B; T; C; q) := ´_1(\frac{q}{BC}\text{"}; B; T)$. Thus, we proved that for jNj , $´_2(\text{"}; B; T; C; q)$ the (" + ¯ + ±)-core of the game $(N; V)$ is non-empty.

For " > $\frac{m^{¤}}{2}$ let us de¯ne $´_2(\text{"}; B; T; C; q) := ´_2(\frac{m^{¤}}{2}; B; T; C; q)$. Then for jNj , $´_2(\text{"}; B; T; C; q)$ again the (" + ¯ + ±)-core of the game $(N; V)$ is non-empty. ∎

8 Appendix 2: Mathematical Examples.

In the following example we enlarge the player set of a given game so that the number of players of each type is arbitrary and illustrate the conditions of Theorem 1.

Example A1. Let $(N; V)$ be a game satisfying comprehensiveness. Suppose jNj = T: We construct a collection of games with T player types and strictly e®ective group sizes bounded by B = T: The games are indexed by m 2 Z_+^T: Given a vector m de¯ne $N_m[t] = f(t; q) : q = 1; ...; m_t g$ and de¯ne $N_m = \overset{S}{} N_m[t]$: Next, de¯ne a characteristic function V_m in the following way. Let S be any coalition in N_m containing no more than one player from each set $N_m[t]$ and let S^0 be a subset of N with the same pro¯le. Formally, let ¿ be a type-consistent 1-to-1 correspondence between S and S^0: De¯ne a payo® set V_{mS} for a coalition S as follows:

$$V_{mS} := V_{¿(S)}$$

That is, any coalition S of players in N_m with the same pro¯le as some coalition S^0 of players in N has the same payo® possibilities as S^0. The function V_m is extended to the remaining coalitions in N_m by superadditivity. Speci¯cally, for any S ½ N_m;

$$V_m(S) := \bigcap_{P(S)} \prod_{S^0 2 P(S)} V_m(S^0)$$

where $P(S)$ is a partition of S with the property that $jS^0 \setminus N[t]j$ · 1 for all members S^0 of the partition: The game $(N_m; V_m)$ satis¯es the condition on the class of games of Theorem 1.

Theorem 1 implies that given " > 0 there is a size of game $´_1(\text{"}; T; B)$ such

that for all possible choices of m, if $kmk = jN_mj$ ¸ $´_1(";T;B)$ then the game $(N_m;V_m)$ has a non-empty "-remainder core. But the theorem implies more; the bound $´_1(";T;B)$ is independent of the initial characteristic function V. To clarify this remark, let $(N;V^0)$ be another game with the same player set as $(N;V)$ but there is no necessary relationship between V and V^0: Then for all possible choices of m, if $kmk = jN_mj$ ¸ $´_1(";T;B)$ then the game $(N_m;V_m^0)$ has a non-empty "-remainder core where V_m^0 is de¯ned from V^0 just as V_m was de¯ned from V.

The following example continues Example A1 and illustrates the application of Theorem 2.

Example A2. Let $¹ > 0$ be real number and let B and T be positive integers. Consider the collection of games $(N_m;V_m)$ de¯ned in Example A1. Given a pro¯le $m \in Z_+^T$ with kmk ¸ $´_1(";T;B)$ consider a superadditive game $(N_m;W_m)$ satisfying the following properties:

$$V_m(S) \text{ ½ } W_m(S) \text{ ½ } V_m(S) + 1_{N_m}¹ \text{ for all } S \text{ ½ } N_m:$$

For the game $(N_m;W_m)$ it may be that none of the players are exact substitutes for each other and it may be that there are increasing returns to group size. The games $(N_m;W_m)$; however, are members of the class $G^0((¹;T);(¹;B))$ and Theorem 2 applies. Given $" > 0$ the bound $´_1(";T;B)$; depending only on $";T;$ and $B;$ has the property that if the game N_m has more than $´_1(";T;B)$ players the "-remainder $2¹$-core of the game is non-empty. Theorem 2 applies uniformly to all games $(N_m;V_m)$ derived from a game $(N;V)$ that has T types of players and strictly e®ective groups bounded in size by B:

The ¯nal example motivates the requirement of some transferability of payo®, and, in this paper, the condition of Theorem 3 that q is greater than zero.

Example A3. The positivity of q. Consider a sequence of games without side payments $(N^m;V^m)_{m=1}^1$ where the mth game has $2m + 1$ players: Suppose that any player alone can earn only 0 units or less. Suppose that any two-player coalition can distribute a total payo® of 2 units in any agreed-upon way, while there is no transferability of payo® between coalitions. Suppose only one- and two-player coalitions are e®ective. Then the game is described by the following parameters: $q = 0;T = 1;B = 2;± = ¯ = 0$: Moreover, the game has per capita bound $C = 1$: Thus the game satis¯es strict small group e®ectiveness and per capita boundedness. However, the $\frac{1}{3}$-core of the game is empty for arbitrarily large values of m: (At any feasible payo® vector, at least one player gets 0 units and some other player no more than 1 unit. These two players can form a coalition and gain $\frac{1}{2}$ each.)

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